Pattern Matching and Fixed Points: Resource Types and Strong Call-By-Need*

Pablo Barenbaum Eduardo Bonelli Kareem Mohamed

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Abstract

Resource types are types that statically quantify some aspect of program execution. They come in various guises; this paper focusses on a manifestation of resource types known as *non-idempotent intersection types*. We use them to characterize weak normalisation for a type-erased lambda calculus for the Calculus of Inductive Construction ($\lambda_{\mathtt{e}}$), as introduced by Gregoire and Leroy. The $\lambda_{\mathtt{e}}$ calculus consists of the lambda calculus together with constructors, pattern matching and a fixed-point operator. The characterization is then used to prove the completeness of a *strong call-by-need* strategy for $\lambda_{\mathtt{e}}$. This strategy operates on *open terms*:

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rather than having evaluation stop when it reaches an abstraction, as in weak call-by-need, it computes *strong* normal forms by admitting reduction inside the body of abstractions and substitutions. Moreover, argument evaluation is by-need: arguments are evaluated when needed and at most once. Such a notion of reduction is of interest in areas such as partial evaluation and proof-checkers such as Coq.

1 Introduction

This paper is about the lambda calculus, the programming language that lies at the core of all Functional Programming Languages (FPL). FPLs evaluate *programs* until a *value* is obtained, if such a value exists at all. Programs are modeled as *closed* lambda calculus terms; *values* are a subset of programs that represent the possible results that one obtains by evaluation, typical examples being numerals, booleans and abstractions. FPLs implement an evaluation strategy called *weak reduction* since evaluation does not take place under an abstraction. When computing values from programs, such strategies typically also implement some form of memoization or *sharing* in order to avoid performing duplicate work. Moreover, this work is only performed if it is actually needed for obtaining a value. One thus speaks of *call-by-need strategies* for weak reduction, a notion that originates in the seminal work of Wadsworth [Wad71].

Often one is interested in reducing *inside* the bodies of abstractions. One simple example is a technique known as *partial evaluation* (PE). In PE one has knowledge about some, but not all, of the arguments to a function, the remaining ones being supplied at a later *stage*. In this case, one can specialize the code of the function to those specific arguments. Here is a classic example. Assume we have a function for computing m^n, $n \geq 0$:

$$\mathit{pow} := \lambda n.\lambda m.\mathrm{if} n = 0 \text{ then } 1 \text{ else } m * \mathit{pow}\ (n-1)\ m$$

If we know the value of n to be 2, then we can produce a more efficient version of $\mathit{pow}\ 2$ as follows:

$$\begin{aligned}
\mathit{pow}\,2 &\to \lambda m.\text{if } 2 = 0 \text{ then } 1 \text{ else } m * (\mathit{pow}\ (2-1)\ m)\\
&\to \lambda m.m * (\mathit{pow}\ (2-1)\ m)\\
&\to \lambda m.m * (\text{if } 1 = 0 \text{ then } 1 \text{ else } m * (\mathit{pow}\ (1-1)\ m))\\
&\to \lambda m.m * (m * (\mathit{pow}\ (1-1)\ m))\\
&\to \lambda m.m * (m * (\text{if } 0 = 0 \text{ then } 1 \text{ else } m * (\mathit{pow}\ (0-1)\ m)))\\
&\to \lambda m.m * (m * 1)\\
&\to \lambda m.m * m
\end{aligned}$$

Notice that all the reduction steps depicted above, take place under the lambda abstraction λm. Such a notion of reduction is called *strong reduction*. The values computed are normal forms which we refer to as *strong* normal forms to distinguish them from the normal forms of weak reduction.

Another area of interest of strong reduction is in the implementation of proof assistants that require checking for definitional equality. Proof assistants, such as Coq, that rely on definitional equality of types typically include a typing rule called *conversion*:

$$\frac{\Gamma \vdash t : \tau \quad \tau \equiv \sigma}{\Gamma \vdash t : \sigma}$$

Checking that types τ and σ are definitionally equal, denoted by the judgement $\tau \equiv \sigma$, involves computing the strong normal form of these types. In turn, this involves computing the strong normal form of the terms that occur in them. The reason that terms occur in types is that the type theory on which such proof assistants are erected are dependent type theories. These terms include constants for building values of inductive types and fixed-point operators for encoding recursive functions over inductive types.

The Extended Lambda Calculus. The Extended Lambda Calculus [GL02] (referred in *op.cit.* as the "type-erased lambda calculus"), denoted $\lambda_{\mathtt{e}}$, extends the lambda calculus with constants, pattern matching and fixed-points. Here is an example of a term in $\lambda_{\mathtt{e}}$ that computes the length of a list encoded with constants **nil** and **cons** (see Sec. 2 for a detailed definition of $\lambda_{\mathtt{e}}$):

$$\mathtt{fix}\,(l.\ \lambda xs.\ \mathtt{case}\ xs\ \mathtt{of}\ (\mathbf{nil} \Rightarrow \mathbf{zero}) \cdot (\mathbf{cons}\, hd\, tl \Rightarrow \mathbf{succ}\,(l\, tl)))$$

The Extended Lambda Calculus is a subset of Gallina, the specification language of Coq. Gregoire and Leroy [GL02] study judicious mechanisms for implementing strong reduction in $\lambda_{\mathtt{e}}$ in order to apply it to check type-conversion. They propose a notion of strong reduction for $\lambda_{\mathtt{e}}$ on open terms (*i.e.* terms possibly containing free variables) called *symbolic call-by-value*. Symbolic CBV iterates call-by-value, accumulating terms for which computation cannot progress. No notion of sharing is addressed. Indeed, unnecessary computation may be performed. For example, consider the following $\lambda_{\mathtt{e}}$ term, where $\mathtt{id}$ abbreviates the identity term $\lambda z.z$:

$$\mathtt{case}\ \mathbf{c}\,(\mathtt{id}\,\mathtt{id})\ \mathtt{of}\ \mathbf{c}\,x \Rightarrow \mathbf{d} \tag{1}$$

This term is a case expression that has *condition* $\mathbf{c}\,(\mathtt{id}\,\mathtt{id})$ and *branch* $\mathbf{c}\,x \Rightarrow \mathbf{d}$, the pattern of the branch being $\mathbf{c}\,x$ and the target $\mathbf{d}$. Notice that the branch does not make use of x in the target. However, Symbolic CBV will contract the redex $\mathtt{id}\,\mathtt{id}$ since the argument of $\mathbf{c}$ must be a value before selecting the matching branch.

Adding Sharing to Strong Reduction. This paper proposes a notion of strong reduction for $\lambda_{\mathtt{e}}$ that only reduces redexes that are needed in order to obtain the (strong) normal form. *e.g.* our strategy will not reduce the $\mathtt{id}\,\mathtt{id}$ redex in (1). Arguments to functions will be suspended until needed. Moreover, they will be reduced at most once. The latter requires us to extend $\lambda_{\mathtt{e}}$ with an additional syntactic construct to hold such suspended arguments: *explicit substitutions*. The resulting theory of sharing ($\lambda_{\mathtt{sh}}$) replaces the usual β rule in the lambda calculus $(\lambda x.t)\, s \to_\beta t\{x := s\}$ with $(\lambda x.t)\, s \to_{\mathtt{dB}} t[x\backslash s]$. The term $t[x\backslash s]$ is also often written $\mathtt{let}\ x = s\ \mathtt{in}\ t$. Notice that rather than substitute all free occurrences of x in t with s, the latter suspends this substitution process. Moreover, since explicit substitutions may now hide redexes, such as in $(\lambda x.x)[x\backslash y]\, z$, a slightly more general formulation of $\mathtt{dB}$ is adopted, namely $(\lambda x.t)\mathtt{L}\, s \to_{\mathtt{dB}} t[x\backslash s]\mathtt{L}$. The notation $\mathtt{L}$ denotes a possibly empty list of explicit substitutions [AK10].

In order to make use of an argument suspended in an explicit substitution it has to have been fully evaluated to a result. As mentioned above, results typically include numerals, booleans and abstractions. In our setting, values shall either be abstractions or terms headed with constants (*cf.* Sec. 2). An additional consideration is that values v may be "polluted" with explicit substitutions $\mathtt{L}$. We thus have the following rule to be able to use a suspended argument: $\mathtt{C}[\![x]\!][x\backslash v\mathtt{L}] \to_{\mathtt{lsv}} \mathtt{C}[v][x\backslash v]\mathtt{L}$. Note how this rule makes use of a context $\mathtt{C}$ and the notation $\mathtt{C}[\![x]\!]$ to mean that there is a free occurrence of x. For example, $(x\,x)[x\backslash \lambda y.y] \to_{\mathtt{lsv}} ((\lambda y.y)\,x)[x\backslash \lambda y.y]$ and also $(x\,x)[x\backslash \lambda y.y] \to_{\mathtt{lsv}} (x\,(\lambda y.y))[x\backslash \lambda y.y]$. Of course, also $((\lambda y.z)\,x)[x\backslash \lambda y.y] \to_{\mathtt{lsv}} ((\lambda y.z)\,(\lambda y.y))[x\backslash \lambda y.y]$, even though x is not needed since it will be discarded by $(\lambda y.z)$. Selecting only needed occurrences of x to be replaced by results will be achieved by imposing a specific *reduction strategy* on $\to_{\mathtt{sh}}$, as described below. Additional rules for dealing with case expressions and fixed-points are discussed in Sec. 2.

Resource Types for the Lambda Calculus. The challenge in establishing that the strong call-by-need theory is well-behaved is proving that every term in $\lambda_{\mathtt{e}}$ that has a normal form *also* has a normal form in $\lambda_{\mathtt{e}}$ *with sharing* ($\lambda_{\mathtt{sh}}$). That is, that the restricted notion of replacement of results is general enough to capture all normalising derivations in $\lambda_{\mathtt{e}}$.

$$t \in \mathsf{WN}(\lambda_{\mathtt{e}}) \Rightarrow t \in \mathsf{WN}(\lambda_{\mathtt{sh}})$$

The notation $\mathsf{WN}(\lambda_{\mathtt{e}})$ denotes the set of $\lambda_{\mathtt{e}}$-terms that have a normal form via $\lambda_{\mathtt{e}}$ and similarly for $\mathsf{WN}(\lambda_{\mathtt{sh}})$. Arguably this has been the main technical hurdle in prior works for weak reduction such as [MOW98, AFM$^+$95] which introduced *ad hoc* notions of development, redex tracking and dags with boxes. It was recently noticed [Kes16] that by devising an appropriate *non-idempotent intersection type system* $\mathcal{T}$ for $\lambda_{\mathtt{sh}}$, one could achieve this as follows:

$$t \in \mathsf{WN}(\lambda_{\mathtt{e}}) \overbrace{\Rightarrow}^{\textit{Step 1}} t \in \mathsf{Typable}(\mathcal{T}) \overbrace{\Rightarrow}^{\textit{Step 2}} t \in \mathsf{WN}(\lambda_{\mathtt{sh}}) \tag{2}$$

Non-idempotent intersection types [Gar94] track/count the uses of variables in terms and thus restrict reduction properties of its typable terms *e.g.*, they may be used to characterize weak, head and strongly normalising terms [BKV17]. If one writes non-idempotent intersection types as multisets of types, then $x : [\tau_1, \tau_2]$ means that x has to be used twice with the indicated types. Similarly, $y : [[\tau_1, \tau_2] \to \tau_3]$ means that y has to be used once and that the argument to y has to be typed twice, once with type τ_1 and once with τ_2.

The argument behind Step 1 is roughly as follows. Given a term in normal form, for any variable x, one "counts" each of its occurrences by giving it a type and then takes the multiset of all those types as the type of x. Then one shows a Subject Expansion result: if the contractum via a reduction step of a term is typable, then so is the term itself. For those variables that reduction does not erase, their type in the contractum can be used to type the redex, those that are erased are not typed at all in the redex (they occur in subterms that are typed with the empty multiset).

The argument behind Step 2 involves showing that reduction of redexes that are typed in $\mathcal{T}$ decreases the size of the type derivation. Reduction of redexes that are not typed could lead to non-termination [Kes16, BBBK17]. For example, $x\,\Omega$, where Ω is $(\lambda x.x\,x)\,(\lambda x.x\,x)$, is typable by setting x to have type $[] \to \alpha$, for α a type variable; the empty multiset of type $[]$ allowing the typing of Ω to *not* be accounted for. However, the term is not normalising in $\lambda_{\mathtt{sh}}$ or any theory of sharing that allows β to be simulated.

Resource Types for the Extended Lambda Calculus. We must adapt this counting technique to the setting of `case` and fixed points. It turns out that the challenge lies in dealing with `case` (however, see Rem. 2). Consider the term:

$$\mathtt{case}\ \mathbf{c}\ \mathtt{of}\ (\mathbf{c} \Rightarrow \mathbf{d}) \cdot (\mathbf{d} \Rightarrow \Omega)$$

It will evaluate to $\mathbf{d}$ and hence should be typable in $\mathcal{T}$ (*cf.* Step 1). Since Ω does not participate at all in computing $\mathbf{d}$, there is no need for $\mathcal{T}$ to account for it. Thus our proposed typing rules will only type branches that are actually used to compute the normal form. This, however, beckons the question of what happens with case expressions that block. In an expression such as:

$$\mathtt{case}\ \mathbf{c}\ \mathtt{of}\ (\mathbf{d} \Rightarrow \mathbf{d}) \cdot (\mathbf{e} \Rightarrow \mathbf{e})$$

all its subexpressions are part of the normal form and hence should be typed. Our proposed typing rule shall ensure this, thus avoiding typing terms such as:

$$\texttt{case}\ \mathbf{c}\ \texttt{of}\ (\mathbf{d} \Rightarrow \mathbf{d}) \cdot (\mathbf{e} \Rightarrow \Omega)$$

where, although matching is blocked, have not strong normal form in $\lambda_{\mathtt{e}}$ or $\lambda_{\mathtt{sh}}$. Since blocked case expressions could be applied to arguments, further considerations are required. Consider the term:

$$(\texttt{case}\ \mathbf{c}\ \texttt{of}\ \mathbf{d} \Rightarrow \mathbf{d})\, \Omega$$

It does not have a normal form in $\lambda_{\mathtt{e}}$ or $\lambda_{\mathtt{sh}}$ and hence should not be typable (Step 2). To ensure that, we need the type assigned to this term to provide access to the types of the arguments to which it is applied, namely Ω, so that constraints on these types may be placed. In other words, we need to devise $\mathcal{T}$ such that it gives $\texttt{case}\ \mathbf{c}\ \texttt{of}\ (\mathbf{d} \Rightarrow \mathbf{d})\, \Omega$ a type that *includes* that of Ω. This would enable us to state conditions that do not allow this term to be typed but do allow a term such as $(\texttt{case}\ \mathbf{c}\ \texttt{of}\ \mathbf{d} \Rightarrow \mathbf{d})\, \mathbf{e}$ to be typed. This motivates our notions of *error type* and *error log* (*cf.* Sec. 3).

The above examples were all closed terms. Open terms pose additional problems. Consider the term:

$$\texttt{case}\ x\ \texttt{of}\ (\mathbf{c} \Rightarrow \mathbf{d}) \cdot (\mathbf{e} \Rightarrow \Omega)$$

Although it does not have a normal form in $\lambda_{\mathtt{sh}}$, it is typable with type $\mathbf{d}$, if $x : [\mathbf{c}]$. Notice, moreover, that the empty multiset of types does not occur in the type of x (in fact, it meets all the requirements of [Kes16, BBBK17]). The reason it is typable is that Ω is never accounted: since x is known to have type $[\mathbf{c}]$, only the $\mathbf{c} \Rightarrow \mathbf{d}$ branch is typed. Hence some restrictions on the types of free variables in Step 2 must be put forward, clearly variables cannot be assigned any type. In particular, it seems we should not allow contant types such as $\mathbf{c}$ to occur *positively* in the types of free variables. Indeed, we will require that constant types do not occur *positively* in the typing context and *negatively* in error logs and in the predicate (*cf.* notion of **good** typing judgements in Sec. 4.1). Note that constants can occur *negatively* in the types of variables. This allows terms such as $x\, \mathbf{c}$ to be typable.

One final consideration is that collecting all the requirements, both on empty multiset types and type constants, should still allow weakly normalising terms in $\lambda_{\mathtt{e}}$ to be typable in $\mathcal{T}$. We will see that this will indeed be the case.

A Strong Call-by-Need Strategy. As mentioned, reduction in the theory of sharing may involve reducing redexes that are not needed. By restricting reduction in $\rightarrow_{\mathtt{sh}}$ to a subset of the contexts where reduction can take place, we can ensure that only needed redexes are reduced. We next illustrate, through an example, our call-by-need strategy. The strategy will be denoted $\rightarrowtail_{\mathtt{sh}}$, "sh" for sharing. Consider the term:

$$(\texttt{case}\ (\lambda y.x\, y)(\texttt{id}\, \texttt{id})\ \texttt{of}\ \mathbf{c} \Rightarrow \mathbf{d})\ (\texttt{id}\, \mathbf{c})$$

It consists of a case expression applied to an argument. This case expression has a *condition* $(\lambda y.x\, y)(\texttt{id}\ \texttt{id})$, a *branch* $\mathbf{c} \Rightarrow \mathbf{d}$ with *pattern* $\mathbf{c}$ and *target* $\mathbf{d}$, and is applied to an argument $\texttt{id}\, \mathbf{c}$. The first reduction step for this term is the same as for weak call-by-need, namely reducing the β-redex $(\lambda y.x\, y)(\texttt{id}\ \texttt{id})$ in the condition of the case. It must be reduced in order to determine which branch, if any, is to be selected. This β-redex is turned into $(x\, y)[y\backslash \texttt{id}\ \texttt{id}]$. The resulting term is:

$$(\texttt{case}\ (x\, y)[y\backslash \texttt{id}\ \texttt{id}]\ \texttt{of}\ \mathbf{c} \Rightarrow \mathbf{d})(\texttt{id}\, \mathbf{c})$$

A weak call-by-need strategy would stop there, since the case expression is stuck. In the strong case, however, reduction should continue to complete the evaluation of the term until a strong normal form is reached. Both the body of the explicit substitution $\mathtt{id}\,\mathtt{id}$ and also the argument of the stuck case expression $\mathtt{id}\,\mathbf{c}$ are needed to produce the strong normal form. Thus evaluation must continue with these redexes. That these redexes are indeed selected and, moreover, which one is selected first, depends on an appropriate notion of *evaluation context*. Our strategy will include an evaluation context $\mathtt{C}$ of the form $(\mathtt{case}\ (x\,y)[y\backslash\Box]\ \mathtt{of}\ \mathbf{c} \Rightarrow \mathbf{d})(\mathtt{id}\,\mathbf{c})$ and hence the body of the explicit substitution will be reduced next. Notice that in order for the focus of computation to be placed in the body of an explicit substitution, its target y should be needed. In this particular case, it is because x is free but y is needed for computing the strong normal form. However, in a term such $\lambda x.\mathbf{c}[y\backslash\mathtt{id}\,\mathtt{id}]$, the β-redex $\mathtt{id}\,\mathtt{id}$ is not needed for the strong normal form and hence will not be selected by the strategy.

The remaining computation steps leading to the strong normal form are depicted below.

$$
\begin{array}{lll}
 & (\mathtt{case}\ (\lambda y.x\,y)(\mathtt{id}\,\mathtt{id})\ \mathtt{of}\ \mathbf{c} \Rightarrow \mathbf{d})(\mathtt{id}\,\mathbf{c}) & \\
\rightarrowtail_{\mathtt{sh}} & (\mathtt{case}\ (x\,y)[y\backslash\mathtt{id}\ \mathtt{id}]\ \mathtt{of}\ \mathbf{c} \Rightarrow \mathbf{d})(\mathtt{id}\,\mathbf{c}) & \\
\rightarrowtail_{\mathtt{sh}} & (\mathtt{case}\ (x\,y)[y\backslash z[z\backslash\mathtt{id}]]\ \mathtt{of}\ \mathbf{c} \Rightarrow \mathbf{d})(\mathtt{id}\,\mathbf{c}) & \\
\rightarrowtail_{\mathtt{sh}} & (\mathtt{case}\ (x\,y)[y\backslash\mathtt{id}[z\backslash\mathtt{id}]]\ \mathtt{of}\ \mathbf{c} \Rightarrow \mathbf{d})(\mathtt{id}\,\mathbf{c}) & \\
\rightarrowtail_{\mathtt{sh}} & (\mathtt{case}\ (x\ \mathtt{id})[y\backslash\mathtt{id}][z\backslash\mathtt{id}]\ \mathtt{of}\ \mathbf{c} \Rightarrow \mathbf{d})(\mathtt{id}\,\mathbf{c}) & (*) \\
\rightarrowtail_{\mathtt{sh}} & (\mathtt{case}\ (x\ \mathtt{id})[y\backslash\mathtt{id}][z\backslash\mathtt{id}]\ \mathtt{of}\ \mathbf{c} \Rightarrow \mathbf{d})(z[z\backslash\mathbf{c}]) & \\
\rightarrowtail_{\mathtt{sh}} & (\mathtt{case}\ (x\ \mathtt{id})[y\backslash\mathtt{id}][z\backslash\mathtt{id}]\ \mathtt{of}\ \mathbf{c} \Rightarrow \mathbf{d})(\mathbf{c}[z\backslash\mathbf{c}]) &
\end{array}
$$

Note that in the fourth step (indicated with an asterisk), y has been replaced by $\mathtt{id}$. As in weak call-by-need, only *answers* shall be substituted for variables. Answers are abstractions under a possibly empty list of explicit substitutions or data structures possibly interspersed with explicit substitutions. Finally, crucial to defining the strong call-by-need strategy will be identifying variables and case expressions that will persist. The former are referred to as *frozen* variables and are free variables (or those that are bound under abstractions and branches of case expressions) that we know will never be substituted by an answer. The latter are referred to as *error terms* and are case expressions that we know will be stuck forever. An example of the former is $x\,y$ in $(x\,y)[y\backslash\mathtt{id}\,\mathtt{id}]$; an example of the latter is $\mathtt{case}\ (x\ \mathtt{id})[y\backslash\mathtt{id}][z\backslash\mathtt{id}]\ \mathtt{of}\ \mathbf{c} \Rightarrow \mathbf{d}$ in $(\mathtt{case}\ (x\ \mathtt{id})[y\backslash\mathtt{id}][z\backslash\mathtt{id}]\ \mathtt{of}\ \mathbf{c} \Rightarrow \mathbf{d})(\mathtt{id}\,\mathbf{c})$.

Contribution. The main contributions of this paper are:

- A non-idempotent intersection type system for $\lambda_{\mathtt{e}}$ satisfying (2).
- A strong call-by-need strategy for $\lambda_{\mathtt{e}}$.
- A proof of completeness of the strategy.

Discussion. Although comparison with related work is developed in Sec. 7, we would like to comment on [BBBK17], the most closely related work (co-authored by two of the present authors). The work in [BBBK17] proposes a strong call-by-need strategy for the lambda calculus *without* matching and fixed point. It should perhaps be mentioned that standard encodings of inductive types in the untyped lambda calculus such as the Church or Scott encodings do not address the above mentioned problems. The culprit is the absence of a high-level construct such as "case" which makes the notion of "blocked case" not obvious in terms of the underlying encoding. Eg. consider the standard Church encoding of a constant $\mathbf{c}_i$ of arity $a(i)$:

$$\lambda x_1 \ldots x_{a(i)}.\lambda c_1 \ldots c_n.c_i\,(x_1\,\vec{c}) \ldots (x_n\,\vec{c})$$

How would the blocked case expression $\texttt{case}\ \mathbf{c}\ \texttt{of}\ (\mathbf{d} \Rightarrow \mathbf{d})$ be encoded? Resorting to the iterators of Church encodings, we would have: $(\lambda cd.c)\ ?\ (\lambda cd.d)$, where the question mark is the missing branch. Consider also $\texttt{case}\ x\ \texttt{of}\ (\mathbf{c} \Rightarrow \mathbf{d}; \mathbf{d} \Rightarrow \Omega)$ with $x : [c]$. This would be encoded as $x\,(\lambda cd.d)\,(\lambda cd.\Omega)$. The non-idempotent intersection type system of [BBBK17] does not account for Ω.

Structure of the paper. We revisit the Extended Lambda Calculus $\lambda_{\mathtt{e}}$ in Sec. 2 and also introduce the theory of sharing for it $\lambda_{\mathtt{sh}}$ in the same section. Sec. 3 introduces the type system $\mathcal{T}$. Sec. 4 addresses Step 1 and 2 as described above. We present the strong call-by-need strategy in Sec. 5 and address the final completeness result (standardization theorem) in Sec. 6. Finally, we comment on related work and conclude, suggesting further avenues to pursue.

2 A Theory of Sharing for the Extended Lambda Calculus

We assume given a denumerable set of variables $x, y, z, \ldots$ and constants $\mathbf{c}, \mathbf{c}', \mathbf{c}'', \ldots$.

Definition 1. *The* **terms of the Extended Lambda Calculus** $\Lambda_{\mathtt{e}}$ *are defined as follows:*

$$\begin{array}{llcl}
\textbf{Terms} & t, s, u, \ldots & ::= & x \mid \lambda x.t \mid t\,s \mid \mathbf{c} \mid \texttt{case}\ t\ \texttt{of}\ \bar{b} \mid \texttt{fix}(x.t) \\
\textbf{Branch} & b & ::= & \mathbf{c}\bar{x} \Rightarrow t \\
\textbf{Contexts} & \mathtt{C} & ::= & \Box \mid \lambda x.\mathtt{C} \mid \mathtt{C}t \mid t\,\mathtt{C} \mid \texttt{fix}(x.\mathtt{C}) \mid \texttt{case}\ \mathtt{C}\ \texttt{of}\ \bar{b} \\
 & & \mid & \texttt{case}\ t\ \texttt{of}\ (\mathbf{c}_1\bar{x}_1 \Rightarrow s_1) \ldots \\
 & & & \quad \ldots (\mathbf{c}_i\bar{x}_i \Rightarrow \mathtt{C}) \ldots (\mathbf{c}_n\bar{x}_n \Rightarrow s_n)
\end{array}$$

In addition to the standard terms of the lambda calculus, **variables**, **abstraction** and **application**, we have **constants**, **case expressions** and **fixed-point expressions**. In $\texttt{case}\ t\ \texttt{of}\ \bar{b}$ we say t is the **condition** of the case and $\bar{b}$ are its **branches**; $\bar{b}$ is shorthand for a (possibly empty) sequence of branches. If $I = \{1, 2, \ldots, n\}$, we sometimes write $\texttt{case}\ t\ \texttt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i \in I}$ for case expressions. Branches are assumed to be syntactically restricted so that if $i \neq j$ then $(\mathbf{c}_i, |\bar{x}_i|) \neq (\mathbf{c}_j, |\bar{x}_j|)$, where $|\bar{x}_j|$ denotes the length of the sequence $\bar{x}_j$. Moreover, the list $\bar{x}_i$ of formal parameters in each branch is assumed to have no repeats. We write $\texttt{fix}(x.t)$ for the standard fixed-point expression. We often write $\lambda\bar{x}.t$ for $\lambda x_1.\ldots\lambda x_n.t$ if $\bar{x}$ is the sequence of variables $x_1 \cdot \ldots \cdot x_n$ and similarly $t\bar{s}$ stands for $ts_1 \ldots s_n$ if $\bar{s} = s_1 \cdot \ldots \cdot s_n$. **Free** and **bound** variables are defined as expected. In particular, x is bound by a fixed point operator $\texttt{fix}(x.t)$, and all the variables $x_1, \ldots, x_n$ are bound in a branch $\mathbf{c}x_1 \ldots x_n \Rightarrow t$. Terms are considered up to renaming of bound variables. A **context** is a term $\mathtt{C}$ with a single free occurrence of a hole $\Box$, and the variable-capturing substitution of the hole $\Box$ by a term t is written $\mathtt{C}[t]$; $\mathtt{C}[\![t]\!]$ has the additional requirement that no free variables in t are bound in $\mathtt{C}$.

Remark 2. *In [GL02] a family of fixed-point operators fix_n, for n a positive integer, is used. The index n indicates the expected number of arguments and also the index of the argument that is used to guard recursion to avoid infinite unfoldings. The type system of the Calculus of Constructions guarantees that the recursive function is applied to strict subterms of the n-th argument. Although we use the more general fixed-point operator fix in our calculus similar ideas to "case" can be applied to fix_n which "blocks" if given less than n arguments.*

Definition 3. *The* **Extended Lambda Calculus** $\lambda_{\mathtt{e}}$ *is given by the following reduction rules* *over $\Lambda_{\mathtt{e}}$, closed by arbitrary contexts. We write $\to_{\mathtt{e}}$ for the resulting reduction relation.*

$$
\begin{array}{rcll}
(\lambda x.t)s & \mapsto_{\mathtt{dB}} & t\{x := s\} & (\beta) \\
\mathtt{fix}(x.t) & \mapsto_{\mathtt{fix}} & t\{x := \mathtt{fix}(x.t)\} & (\mathit{fix}) \\
\mathtt{case}\ \mathbf{c}_j\bar{t}\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} & \mapsto_{\mathtt{case}} & s_j\{\bar{x}_j := \bar{t}\} & (\mathit{case}) \\
 & & \textit{if } j \in I \textit{ and } |\bar{t}| = |\bar{x}_j| &
\end{array}
$$

Capture-avoiding substitution of a variable x by a term s in a term t is written $t\{x := s\}$. Similarly, the simultaneous capture-avoiding substitution of a list of variables $\bar{x}$ by a list of terms $\bar{s}$ of the same length in a term t is written $t\{\bar{x} := \bar{s}\}$. A term t **matches** with a branch $\mathbf{c}\bar{x} \Rightarrow s$ if $t = \mathbf{c}\bar{s}$ with $|\bar{s}| = |\bar{x}|$. A term t matches with a list of branches if it matches with at least one branch. Given our syntactic formation condition on case-expressions, in $\lambda_{\mathtt{e}}$ terms in fact match with at most one branch. Note that term reduction may become blocked if the condition of a case does not match any branch (and never will). The normal forms of $\lambda_{\mathtt{e}}$ may be characterized as follows:

Lemma 4 (Normal forms)**.** *The normal forms of $\lambda_{\mathtt{e}}$ are characterized by the grammar:*

$$
N \quad ::= \quad \lambda\bar{x}.x\bar{N} \mid \lambda\bar{x}.\mathbf{c}\bar{N} \mid \lambda\bar{x}.(\mathtt{case}\ N_0\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow N_i)_{i\in I})\bar{N}
$$

where N_0 does not match with $(\mathbf{c}_i\bar{x}_i \Rightarrow N_i)_{i\in I}$. Note that the lists $\bar{x}$ and $\bar{N}$ may be empty.

Proof. By structural induction on the set of terms.

- x. The first case applies where $\bar{x}$ is empty and so is $\bar{N}$.
- $\lambda x.t$. Then t must be in normal form too, the *i.h.* applies and we conclude from that.
- $t\,s$. Then both t and s are in normal form and we resort to the *i.h.* on each to obtain $t = N_1$ and $s = N_2$. Moreover, t is not an abstraction so it must be one of the following:
  - $x\bar{N}$. Then $t\,s = x\bar{N}\,N_2$ and we conclude.
  - $\mathbf{c}\bar{N}$. Then $t\,s = \mathbf{c}\bar{N}\,N_2$ and we conclude.
  - $(\mathtt{case}\ N_0\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow N_i)_{i\in I})\bar{N}$, where N_0 does not match with $(\mathbf{c}_i\bar{x}_i \Rightarrow N_i)_{i\in I}$. Then $t\,s = (\mathtt{case}\ N_0\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow N_i)_{i\in I})\bar{N}\,N_2$ and we conclude.
- $\mathtt{fix}(x.t)$. This case is not possible since the term is not in normal form.
- $\mathbf{c}$. The second case applies where $\bar{x}$ is empty and so is $\bar{N}$.
- $\mathtt{case}\ t\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ with t and $(s_i)_{i\in I}$, all in normal form. We apply the *i.h.* to these terms and deduce $t = N_0$ and $(s_i)_{i\in I} = (N_i)_{i\in I}$. Note that N_0 does not match the branches $(\mathbf{c}_i\bar{x}_i \Rightarrow N_i)_{i\in I}$ for otherwise the original term would not have been in normal form. Therefore, we conclude.

□

Remark 5. *From Lem. 4 every normal form is in one of the forms: $\lambda x.N$, $x\bar{N}$, $\mathbf{c}\bar{N}$ or $(\mathtt{case}\ N_0\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow N_i)_{i\in I})\bar{N}$, where N_0 does not match with $(\mathbf{c}_i\bar{x}_i \Rightarrow N_i)_{i\in I}$.*

Remark 6. *Since we work in an untyped setting blocked terms such as* $\texttt{case}\ \mathbf{c}\ \texttt{of}\ \mathbf{d} \Rightarrow \mathbf{e}$ *must be admitted. In the Calculus of Inductive Constructions, case analysis must be exhaustive.*

We conclude this section with some standard terminology on rewrite systems. Given a notion of reduction $\mathcal{R}$ over a set of terms, we use the following rewriting concepts. A term t is in **$\mathcal{R}$-normal form** ($\mathcal{R}$-nf) if there is no s such that $t \rightarrow_{\mathcal{R}} s$. We write $\mathtt{n}_{\mathcal{R}}$ for any term in $\mathcal{R}$-normal form. We write $\twoheadrightarrow_{\mathcal{R}}$ for the reflexive and transitive closure of any reduction relation $\rightarrow_{\mathcal{R}}$. A term t is **weakly $\mathcal{R}$-normalising**, if there exists s in $\mathcal{R}$-normal form s.t. $t \twoheadrightarrow_{\mathcal{R}} s$. $\mathtt{NF}(\rightarrow_{\mathcal{R}})$ denotes the set of $\mathcal{R}$-normal forms and $\mathtt{WN}(\rightarrow_{\mathcal{R}})$ the set of weakly $\mathcal{R}$-normalising terms. We write $\leftrightarrow^*_{\mathcal{R}}$ for the reflexive, symmetric, transitive closure of $\rightarrow_{\mathcal{R}}$. We say that t is **definable as** s in $\rightarrow_{\mathcal{R}}$, if $t \leftrightarrow^*_{\mathcal{R}} s$ for $s \in \mathtt{NF}(\mathcal{R})$. Also, t is **definable** in $\rightarrow_{\mathcal{R}}$ if it is definable as s, for some s, in $\rightarrow_{\mathcal{R}}$. We use ":=" for definitional equality.

Remark 7. *t is definable in $\lambda_{\mathtt{e}}$ iff $t \in \mathtt{WN}(\rightarrow_{\mathtt{e}})$. This follows from confluence of $\rightarrow_{\mathtt{e}}$.*

2.1 A Theory of Sharing

Definition 8. *The* **terms of the theory of sharing terms** $\Lambda_{\mathtt{sh}}$ *are defined as follows:*

$$t, s, u, \ldots \quad ::= \quad x \mid \lambda x.t \mid t\, s \mid \mathtt{fix}(x.t) \mid \mathbf{c} \mid \mathtt{case}\ t\ \mathtt{of}\ \bar{b} \mid t[x\backslash s]$$

A term $t[x\backslash s]$ is called a **closure**, and $[x\backslash s]$ is called an **explicit substitution**. Terms without explicit substitutions are called **pure terms**. Closures are often written as $\mathtt{let}\ x\ \mathtt{be}\ s\ \mathtt{in}\ t$ in the literature (*e.g.* [AFM+95]). The notions of **free** and **bound variables** of extended terms are defined as usual, in particular, $\mathsf{fv}(t[x\backslash s]) = (\mathsf{fv}(t) \setminus \{x\}) \cup \mathsf{fv}(s)$ and $\mathsf{bv}(t[x\backslash s]) = \mathsf{bv}(t) \cup \{x\} \cup \mathsf{bv}(s)$.

Definition 9. *A pure term $t^\diamond$ is obtained from any $t \in \Lambda_{\mathtt{sh}}$ by* **unsharing***:*

$$\begin{array}{rcl@{\qquad}rcl}
x^\diamond & := & x & \mathtt{fix}(x.t)^\diamond & := & \mathtt{fix}(x.t^\diamond) \\
\mathbf{c}^\diamond & := & \mathbf{c} & (\mathtt{case}\ t\ \mathtt{of}\ \bar{b})^\diamond & := & \mathtt{case}\ t^\diamond\ \mathtt{of}\ \bar{b}^\diamond \\
(\lambda x.t)^\diamond & := & \lambda x.t^\diamond & (t[x\backslash s])^\diamond & := & t^\diamond\{x := s^\diamond\} \\
(t\, s)^\diamond & := & t^\diamond\, s^\diamond & (\mathbf{c}\bar{x} \Rightarrow t)^\diamond & := & \mathbf{c}\bar{x} \Rightarrow t^\diamond
\end{array}$$

e.g. $((\mathtt{case}\ z\ \mathtt{of}\ \mathbf{c} \Rightarrow z)[z\backslash \mathbf{d}\,\mathbf{d}])^\diamond = \mathtt{case}\ \mathbf{d}\,\mathbf{d}\ \mathtt{of}\ \mathbf{c} \Rightarrow \mathbf{d}\,\mathbf{d}$. Additional syntactic categories will be required for describing reduction in $\lambda_{\mathtt{sh}}$. First of all, in call-by-need computation one cannot substitute arbitrary terms for variables, rather one substitutes values for variables. In our theory of sharing apart from abstractions as values we also have terms headed by constants as values. Also, values may be embraced by pending explicit substitutions. This leads to the definition of *answers.*

Answers	a	$::=$	$\mathtt{L}[v]$
Values	v	$::=$	$\lambda x.t \mid \mathtt{A}[\mathbf{c}]$
Constant Context	$\mathtt{A}$	$::=$	$\Box \mid \mathtt{L}[\mathtt{A}]\, t$
Substitution Context	$\mathtt{L}$	$::=$	$\Box \mid \mathtt{L}[x\backslash t]$

A term of the form $\mathtt{L}[v]$ is sometimes written $v\mathtt{L}$ and called an **answer**. An answer of the form $(\lambda x.t)\mathtt{L}$ is an **abstraction answer** and one of the form $\mathtt{A}[\mathbf{c}]\mathtt{L}$ is a **constant answer**. An example of the latter is $((\mathbf{c}\, x)[x\backslash y]\, \mathbf{d})[y\backslash s]$.

Second, reduction in $\lambda_{\mathtt{sh}}$ will take place under arbitrary contexts. We define such a set of full contexts next:

$$
\begin{array}{llll}
\textbf{Full Context} & \mathtt{C} & ::= & \Box \mid \lambda x.\mathtt{C} \mid \mathtt{C}\,t \mid t\,\mathtt{C} \mid \mathtt{fix}(x.\mathtt{C}) \\
 & & \mid & \mathtt{case}\ \mathtt{C}\ \mathtt{of}\ \bar{b} \\
 & & \mid & \mathtt{case}\ t\ \mathtt{of}\ (\mathbf{c}_1\bar{x}_1 \Rightarrow s_1)\ldots \\
 & & & \ldots(\mathbf{c}_i\bar{x}_i \Rightarrow \mathtt{C})\ldots(\mathbf{c}_n\bar{x}_n \Rightarrow s_n) \\
 & & \mid & \mathtt{C}[x\backslash s] \mid t[x\backslash \mathtt{C}]
\end{array}
$$

Definition 10. *The* **theory of sharing** $\lambda_{\mathtt{sh}}$ *consists of the reduction rules over* $\Lambda_{\mathtt{sh}}$ *given below, closed by full contexts. We write* $\to_{\mathtt{sh}}$ *for the reduction relation.*

$$
\begin{array}{rll}
(\lambda x.t)\mathtt{L}\,s & \mapsto_{\mathtt{dB}} & t[x\backslash s]\mathtt{L} \\
\mathcal{C}[\![x]\!][x\backslash v\mathtt{L}] & \mapsto_{\mathtt{lsv}} & \mathcal{C}[v][x\backslash v]\mathtt{L} \\
t[x\backslash s] & \mapsto_{\mathtt{gc}} & t, \quad \textit{if } x \notin \mathsf{fv}(t) \\
\mathtt{fix}(x.t) & \mapsto_{\mathtt{fix}} & t[x\backslash \mathtt{fix}(x.t)] \\
\mathtt{case}\ \mathtt{A}[\mathbf{c}_j]\mathtt{L}\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} & \mapsto_{\mathtt{case}} & s_j[\bar{x}_j\backslash \mathtt{A}]\mathtt{L} \\
 & & \textit{if } |\mathtt{A}| = |\bar{x}_j| \textit{ and } j \in I
\end{array}
$$

The $\mathtt{dB}$ rule transforms an application of an abstraction (possibly under multiple explicit substitutions) to an argument, into the body of the abstraction t subject to a new explicit substitution $[x\backslash s]$. The $\mathtt{lsv}$ rule substitutes a free occurrence of x with the value v. Since variables in v might be bound in $\mathtt{L}$ and we do not wish to duplicate $\mathtt{L}$, the scope of the substitution context is adjusted. This rule is said to operate *at a distance* since the explicit substitution is not required to propagate to variables before it is executed [AK10]. It is closely related with the notion of *linear head reduction* [AC17]. Rule $\mathtt{gc}$ removes garbage substitutions. Rule $\mathtt{fix}$ is standard. Rule $\mathtt{case}$ tests whether the condition of the case "matches" one of its branches. Note that the condition $\mathtt{A}[\mathbf{c}_j]\mathtt{L}$ may have explicit substitutions interspersed. The **length** of a constant context is defined as follows: $|\Box| := 0$ and $|\mathtt{L}[\mathtt{A}]\,t| := 1 + |\mathtt{A}|$. Given a list of variables $\bar{x}$ and a constant context $\mathtt{A}$ s.t. their lengths coincide, we define the substitution context $[\bar{x}\backslash \mathtt{A}]$ as follows: $[\epsilon\backslash\Box] := \Box$ and $[\bar{x}, y\backslash \mathtt{L}[\mathtt{A}]\,t] := [\bar{x}\backslash \mathtt{A}]\mathtt{L}[y\backslash t]$. The reduct of $\mapsto_{\mathtt{case}}$ uses this notion to build an appropriate list of explicit substitutions for each parameter of the branch.

Remark 11. *t definable in $\lambda_{\mathtt{sh}}$ iff $t \in \mathtt{WN}(\to_{\mathtt{sh}})$. This follows from confluence of $\to_{\mathtt{sh}}$.*

A characterization of the $\to_{\mathtt{sh}}$-normal forms is given in Fig. 1. The **normal form judgement** $t \in \mathcal{N}$ is defined simultaneously with four other judgements, namely **constant normal forms** $t \in \mathcal{K}$, **structure normal forms** $t \in \mathcal{S}$, **error normal forms** $t \in \mathcal{E}$, and **abstraction normal forms** $t \in \mathcal{L}$. We comment on some salient rules. First note that rule eNfStrt captures a blocked case where its condition is not a blocked case itself. If the condition of the case is $t \in \mathcal{L} \cup \mathcal{S}$, then we know that it cannot possibly match any branch. If $t \in \mathcal{K}$, we must make sure of this, as explained next. We say a term t **enables** a branch in a list of branches $(\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$, written $t \succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$, if $t = \mathtt{A}[\mathbf{c}_j]\mathtt{L}$, for some $\mathtt{A}, \mathtt{L}$, and $j \in I$ and $|\mathtt{A}| = |\bar{x_j}|$. This is the natural extension of the notion of t matching a branch in $\lambda_{\mathtt{e}}$ but where t may be "polluted" with explicit substitutions. Note that if $t \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$, then either, 1) $t \neq \mathtt{A}[\mathbf{c}]\mathtt{L}$ for any $\mathtt{A}, \mathbf{c}, \mathtt{L}$; or 2) $t = \mathtt{A}[\mathbf{c}]\mathtt{L}$ with $\mathbf{c} \notin \{\mathbf{c}_i\}_{i\in I}$; or 3) $t = \mathtt{A}[\mathbf{c}]\mathtt{L}$ and $\mathbf{c} = \mathbf{c}_j$ for some $j \in I$ but $|\mathtt{A}| \neq |\bar{x_j}|$. Rule nfSub is actually a rule scheme in which $\mathcal{X}$ can be any of $\mathcal{S}$, $\mathcal{L}$, $\mathcal{E}$, or $\mathcal{K}$. Condition $x \in \mathsf{fv}(t)$ is required since we would otherwise have a $\mathtt{gc}$-redex. Condition $s \in \mathcal{S} \cup \mathcal{E}$ is required too since we would otherwise have a $\mathtt{lsv}$ redex.

Further insight into the fine structure of the terms in each of these judgements is given by Lem. 13. A description of this fine structure requires identifying so called weak structures and

Figure 1 The set of $\to_{\mathtt{sh}}$-normal forms ($\mathbb{X} \in \{\mathcal{S}, \mathcal{L}, \mathcal{E}, \mathcal{K}\}$)

$$\frac{}{\mathbf{c} \in \mathcal{K}}\ \textsc{cNfCons} \qquad \frac{t \in \mathcal{K} \quad s \in \mathcal{N}}{t\,s \in \mathcal{K}}\ \textsc{cNfApp}$$

$$\frac{}{x \in \mathcal{S}}\ \textsc{sNfVar} \qquad \frac{t \in \mathcal{S} \quad s \in \mathcal{N}}{t\,s \in \mathcal{S}}\ \textsc{sNfApp}$$

$$\frac{t \in \mathcal{K} \cup \mathcal{L} \cup \mathcal{S} \quad t \not\succ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \quad (s_i \in \mathcal{N})_{i \in I}}{\mathtt{case}\ t\ \mathtt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \in \mathcal{E}}\ \textsc{eNfStrt}$$

$$\frac{t \in \mathcal{E} \quad s \in \mathcal{N}}{t\,s \in \mathcal{E}}\ \textsc{eNfApp} \qquad \frac{t \in \mathcal{E} \quad (s_i \in \mathcal{N})_{i \in I}}{\mathtt{case}\ t\ \mathtt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \in \mathcal{E}}\ \textsc{eNfCase}$$

$$\frac{t \in \mathcal{N}}{\lambda x.t \in \mathcal{L}}\ \textsc{lNfLam} \qquad \frac{t \in \mathbb{X} \quad s \in \mathcal{S} \cup \mathcal{E} \quad x \in \mathsf{fv}(t)}{t[x \backslash s] \in \mathbb{X}}\ \textsc{nfSub}$$

$$\frac{t \in \mathcal{K}}{t \in \mathcal{N}}\ \textsc{nfCons} \quad \frac{t \in \mathcal{S}}{t \in \mathcal{N}}\ \textsc{nfStruct} \quad \frac{t \in \mathcal{E}}{t \in \mathcal{N}}\ \textsc{nfError} \quad \frac{t \in \mathcal{L}}{t \in \mathcal{N}}\ \textsc{nfLam}$$

weak error terms. A term of the form $\mathtt{E}[x]$ is called a **weak structure** and a term of the form $\mathtt{F}[\mathtt{case}\ s\ \mathtt{of}\ \bar{b}]$, with s an answer or weak structure and $s \not\succ \bar{b}$, is called a **weak error term**, where contexts $\mathtt{E}$ and $\mathtt{F}$ are defined as follows:

Weak Context	$\mathtt{E}$	$::=$	$\Box \mid \mathtt{E}\,t \mid \mathtt{E}[\![x]\!][x \backslash \mathtt{E}] \mid \mathtt{E}[x \backslash t]$
Weak Error Context	$\mathtt{F}$	$::=$	$\Box \mid \mathtt{F}\,t \mid \mathtt{E}[\![x]\!][x \backslash \mathtt{F}] \mid \mathtt{F}[x \backslash t] \mid \mathtt{case}\ \mathtt{F}\ \mathtt{of}\ \bar{b}$

Lemma 12 (Weak structures and weak error terms are not answers)**.** *$\mathtt{E}[x]$ is not an answer, for any variable x. Neither is $\mathtt{F}[\mathtt{case}\ s\ \mathtt{of}\ \bar{b}]$, for any answer or weak structure s and branches $\bar{b}$ s.t. $s \not\succ \bar{b}$.*

Proof. We first prove that $\mathtt{E}[x]$ is not an answer, by induction on the size of $\mathtt{E}$. If $\mathtt{E} = \Box$, the result is immediate. We consider the inductive cases:

- $\mathtt{E} = \mathtt{E}_1\,t$. From the *i.h.* we know that $\mathtt{E}_1[x]$ is not an answer and, in particular, not a constant answer. Hence neither is $\mathtt{E}[x]$.
- $\mathtt{E} = \mathtt{E}_1[\![y]\!][y \backslash \mathtt{E}_2]$. From the *i.h.* on $\mathtt{E}_1$ we know that $\mathtt{E}_1[\![y]\!]$ is not an answer. Hence neither is $\mathtt{E}[x]$.
- $\mathtt{E} = \mathtt{E}_1[y \backslash t]$. From the *i.h.* on $\mathtt{E}_1$ we know that $\mathtt{E}_1[x]$ is not an answer. Hence neither is $\mathtt{E}[x]$.

The proof that $\mathtt{F}[\mathtt{case}\ s\ \mathtt{of}\ \bar{b}]$, is not an answer is by induction on $\mathtt{F}$ and uses the previous item. $\Box$

Lemma 13 (Characterization of Sharing Normal Forms)**.** *1. $t \in \mathcal{K}$ iff $t = \mathtt{A}[\mathbf{c}]\mathtt{L}$ and $t \in \mathsf{NF}(\to_{\mathtt{sh}}$).*

2. $t \in \mathcal{S}$ *iff* $t = \mathtt{E}[x]$ *and* $t \in \mathsf{NF}(\to_{\mathtt{sh}})$.

3. $t \in \mathcal{E}$ *iff* $t = \mathtt{F}[\mathtt{case}\ s\ \mathtt{of}\ \bar{b}]$, *$s$ is an answer or a weak structure,* $s \not\succ \bar{b}$ *and* $t \in \mathsf{NF}(\to_{\mathtt{sh}})$.

4. $t \in \mathcal{L}$ *iff* $t = (\lambda x.s)\mathtt{L}$ *and* $t \in \mathsf{NF}(\to_{\mathtt{sh}})$.

5. $t \in \mathcal{N}$ *iff* $t \in \mathsf{NF}(\to_{\mathtt{sh}})$.

Proof. For the "only if" direction we proceed by simultaneous induction on the derivations of $t \in \mathcal{K}$, $t \in \mathcal{S}$, $t \in \mathcal{E}$, $t \in \mathcal{L}$ and $t \in \mathcal{N}$.

- cNfCons. Immediate.

- cNfApp. The derivation ends in:

$$\frac{u \in \mathcal{K} \quad s \in \mathcal{N}}{u\, s \in \mathcal{K}}\ \textsc{cNfApp}$$

  The *i.h.* w.r.t. $u \in \mathcal{K}$ gives us $\mathtt{A}', \mathtt{L}'$ and $\mathbf{c}'$ s.t. $u = \mathtt{A}'[\mathbf{c}']\mathtt{L}'$ and $u \in \mathsf{NF}(\to_{\mathtt{sh}})$. We set $\mathtt{A} = \mathtt{A}'\mathtt{L}'\, s$ and $\mathtt{L} = \epsilon$. The *i.h.* w.r.t. $s \in \mathcal{N}$ implies $s \in \mathsf{NF}(\to_{\mathtt{sh}})$. Since u is not an abstraction answer, $u\, s$ is in $\to_{\mathtt{sh}}$-normal form.

- sNfVar. Immediate.

- sNfApp. The derivation ends in:

$$\frac{u \in \mathcal{S} \quad s \in \mathcal{N}}{u\, s \in \mathcal{S}}\ \textsc{sNfApp}$$

  The *i.h.* on $s \in \mathcal{N}$ yields $s \in \mathsf{NF}(\to_{\mathtt{sh}})$. The *i.h.* on $u \in \mathcal{S}$ yields $u \in \mathsf{NF}(\to_{\mathtt{sh}})$ and the existence of $\mathtt{E}'$ s.t. $u = \mathtt{E}'[x]$. We set $\mathtt{E} = \mathtt{E}'\, s$. Since $\mathtt{E}'[x]$ is not an abstraction answer, $\mathtt{E}'[x]\, s \in \mathsf{NF}(\to_{\mathtt{sh}})$.

- eNfStrt. The derivation ends in:

$$\frac{s \in \mathcal{K} \cup \mathcal{L} \cup \mathcal{S} \quad t \not\succ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \quad (u_i \in \mathcal{N})_{i \in I}}{\mathtt{case}\ s\ \mathtt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow u_i)_{i \in I} \in \mathcal{E}}\ \textsc{eNfStrt}$$

  The *i.h.* on $s \in \mathcal{K} \cup \mathcal{L} \cup \mathcal{S}$, $s \in \mathsf{NF}(\to_{\mathtt{sh}})$. The *i.h.* on each $(u_i)_{i \in I}$ yields $u_i \in \mathsf{NF}(\to_{\mathtt{sh}})$, for each $i \in I$. That $t \in \mathsf{NF}(\to_{\mathtt{sh}})$ follows from the hypothesis $t \not\succ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I}$. We set $\mathtt{F} := \Box$ and conclude.

- eNfApp. The derivation ends in:

$$\frac{s \in \mathcal{E} \quad u \in \mathcal{N}}{s\, u \in \mathcal{E}}\ \textsc{eNfApp}$$

  The *i.h.* on $s \in \mathcal{E}$ yields $s \in \mathsf{NF}(\to_{\mathtt{sh}})$ and $s = \mathtt{F}_1[\mathtt{case}\ r\ \mathtt{of}\ \bar{b}']$ and $r \in \mathcal{L} \cup \mathcal{K} \cup \mathcal{S}$ and $r \not\succ \bar{b}'$. The *i.h.* on $u \in \mathcal{N}$ yields $u \in \mathsf{NF}(\to_{\mathtt{sh}})$. We set $\mathtt{F} = \mathtt{F}_1\, u$ and conclude from that $t \in \mathsf{NF}(\to_{\mathtt{sh}})$ from Lem. 12.

- eNfCase. The derivation ends in:

$$\frac{s \in \mathcal{E} \quad (u_i \in \mathcal{N})_{i \in I}}{\texttt{case}\ s\ \texttt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow u_i)_{i \in I} \in \mathcal{E}}\ \text{eNfCase}$$

  The *i.h.* on $s \in \mathcal{E}$ yields $s \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$ and $s = \mathtt{F}_1[\texttt{case}\ r\ \texttt{of}\ \bar{b}']$ and $r \in \mathcal{L} \cup \mathcal{K} \cup \mathcal{S}$ and $r \not\succ \bar{b}'$. The *i.h.* on $u_i \in \mathcal{N}$, for each $i \in I$, yields $u_i \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$. We set $\mathtt{F} = \texttt{case}\ \mathtt{F}_1\ \texttt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow u_i)_{i \in I}$ and conclude that $t \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$ from Lem. 12.

- lNfLam. The derivation ends in:

$$\frac{s \in \mathcal{N}}{\lambda x.s \in \mathcal{L}}\ \text{lNfLam}$$

  By the *i.h.*, $s \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$. We set $\mathtt{L} = \epsilon$ and conclude.

- nfSub. The derivation ends in:

$$\frac{u \in \mathbb{X} \quad s \in \mathcal{S} \cup \mathcal{E} \quad x \in \mathsf{fv}(u)}{u[x \backslash s] \in \mathbb{X}}\ \text{nfSub}$$

  The *i.h.* on $s \in \mathcal{S} \cup \mathcal{E}$, we know either $s = \mathtt{E}[x]$ or $s = \mathtt{E}[\texttt{case}\ s'\ \texttt{of}\ \bar{b}]$ and $s' \in \mathcal{L} \cup \mathcal{K}$ and $s' \not\succ \bar{b}$ and $s \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$. In any case, by Lem. 12, s is not an answer. Hence $u[x \backslash s]$ is not an $\mathtt{lsv}$-redex. We next consider three cases depending on the value of $\mathbb{X}$.

  - $\mathbb{X} = \mathcal{K}$. By the *i.h.* on $u \in \mathcal{K}$, $u = \mathtt{A}'[\mathbf{c}]\mathtt{L}'$ and $u \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$. We set $\mathtt{A} = \mathtt{A}'$ and $\mathtt{L} = \mathtt{L}'[x \backslash s]$.
  - $\mathbb{X} = \mathcal{S}$. By the *i.h.* on $u \in \mathcal{S}$, $u = \mathtt{E}'[x]$ and $u \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$. We set $\mathtt{E} = \mathtt{E}'[x \backslash s]$ and conclude.
  - $\mathbb{X} = \mathcal{E}$. By the *i.h.* on $u \in \mathcal{E}$, $u = \mathtt{F}'[\texttt{case}\ r\ \texttt{of}\ \bar{b}]$ and $r \in \mathcal{L} \cup \mathcal{K} \cup \mathcal{S}$ and $r \not\succ \bar{b}$ and $u \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$. We set $\mathtt{F} = \mathtt{F}'[x \backslash s]$ and conclude.
  - $\mathbb{X} = \mathcal{L}$. By the *i.h.* on $u \in \mathcal{L}$, $u = (\lambda y.r)\mathtt{L}'$ and $u \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$. We set $\mathtt{L} = \mathtt{L}'[x \backslash s]$ and conclude.

- nfCons, nfStruct, nfError, and nfLam. Immediate from the *i.h.*

For the "if" direction we proceed by induction on t. Suppose $t \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$.

- $t = x$. We set $\mathtt{E} := \Box$. $\mathtt{E}[x] = x \in \mathcal{S}$ is immediate from sNfVar. Hence also $x \in \mathcal{N}$, from nfStruct. Thus we prove items 2 and 5.

- $t = \lambda x.s$. We know that $s \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$. By the *i.h.* $s \in \mathcal{N}$. Item 3 follows from lNfLam; item 5 further follows from nfLam.

- $t = t_1\, t_2$. We know $t_1 \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$ and $t_2 \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$. By the *i.h.* w.r.t. the fifth item, $t_1 \in \mathcal{N}$ and $t_2 \in \mathcal{N}$. We now consider three cases:

  - Suppose $t_1\, t_2 = \mathtt{A}[\mathbf{c}]\mathtt{L}$ for some $\mathtt{A}, \mathbf{c}$ and $\mathtt{L}$. Then $\mathtt{L} = \epsilon$ and $\mathtt{A} = \mathtt{A}'\mathtt{L}'\, t_2$ with $\mathtt{A}'[\mathbf{c}]\mathtt{L}' = t_1$. The *i.h.* w.r.t. the first item indicates that $t_1 \in \mathcal{K}$. We conclude that $t_1\, t_2 \in \mathcal{K}$ by using cNfApp.

- Suppose $t_1 t_2 = \mathtt{E}[x]$. Then $\mathtt{E} = \mathtt{E}'\, t_2$ and $t_1 = \mathtt{E}'[x]$. From the *i.h.* w.r.t. the second item we obtain $t_1 \in \mathcal{S}$ and conclude that $t_1\, t_2 \in \mathcal{S}$ from $t_2 \in \mathcal{N}$ and sNfApp.
- Suppose $t_1\, t_2 = \mathtt{F}[\mathtt{case}\ s\ \mathtt{of}\ \bar{b}]$ and $s \in \mathcal{L} \cup \mathcal{K} \cup \mathcal{S}$ and $s \not\succ \bar{b}$, we proceed similarly, noting that the target of every branch in $\bar{b}$ must be in $\mathcal{N}$ given that these branches are in $\mathsf{NF}(\rightarrow_{\mathtt{sh}})$ and that we can resort to the *i.h.* w.r.t. the fifth item.
- The case $t_1\, t_2 = (\lambda x.s)\mathtt{L}$ does not hold and hence item 3 holds trivially.

- $t = \mathtt{fix}(x.s)$. This case is not possible since $t \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$.
- $t = \mathbf{c}$. We set $\mathtt{A} := \Box$ and conclude from cNfCons. This concludes item 1; item 5 then follows from nfCons.
- $t = \mathtt{case}\ s\ \mathtt{of}\ \bar{b}$. Since $t \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$, $s \not\succ \bar{b}$. Moreover, by *i.h.* w.r.t. item 5, $s \in \mathcal{N}$ and each $r_i \in \mathcal{N}$. From the former, $s \in \mathcal{L} \cup \mathcal{S} \cup \mathcal{E} \cup \mathcal{K}$. If $s \in \mathcal{E}$, then we conclude using eNfCase, if $s \in \mathcal{L} \cup \mathcal{S} \cup \mathcal{K}$, then we conclude from eNfStrt.
- $t = t_1[x\backslash t_2]$. Since $t \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$, it must be the case that $x \in \mathsf{fv}(t_1)$ and t_2 is not an answer. Moreover, from $t_1, t_2 \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$ and the *i.h.* w.r.t. the fifth item, $t_1, t_2 \in \mathcal{N}$ and hence $t_1, t_2 \in \mathcal{L} \cup \mathcal{S} \cup \mathcal{K} \cup \mathcal{E}$. Since t_2 is not an answer, $t_2 \in \mathcal{S} \cup \mathcal{E}$. We conclude from nfSub.

$\Box$

We conclude with a simple result that relates reduction in $\rightarrow_{\mathtt{sh}}$ with that in $\rightarrow_{\mathtt{e}}$.

Lemma 14. *Let $t, s \in \Lambda_{\mathtt{sh}}$.*

1. *If $t \rightarrow_{\mathtt{sh}} s$, then $t^\diamond \twoheadrightarrow_{\mathtt{e}} s^\diamond$.*
2. *$t \in \mathtt{NF}(\rightarrow_{\mathtt{sh}})$ implies $t^\diamond \in \mathtt{NF}(\rightarrow_{\mathtt{e}})$.*

3 Intersection Types for the Theory of Sharing

This section introduces $\mathcal{T}$, a non-idempotent intersection type system for $\lambda_{\mathtt{sh}}$. We assume $\alpha, \beta, \gamma, \ldots$ to range over a set of **type variables**. The set of **types** is ranged over by $\tau, \sigma, \rho, \ldots$, and **finite multisets of types** are ranged over by $\mathcal{M}, \mathcal{N}, \mathcal{P}, \ldots$. The empty multiset is written $[]$, and $[\tau_1, \ldots, \tau_n]$ stands for the multiset containing each of the types τ_i with their corresponding multiplicities. Moreover, $\mathcal{M} + \mathcal{N}$ stands for the (additive) union of multisets. For instance $[\mathbf{a}, \mathbf{b}] + [\mathbf{b}, \mathbf{c}] = [\mathbf{a}, \mathbf{b}, \mathbf{b}, \mathbf{c}]$.

Definition 15. *The types of $\mathcal{T}$ are defined as follows:*

Types	τ	$::=$	$\alpha \mid \mathcal{M} \to \tau \mid D \mid E$
Datatypes	D	$::=$	$\mathbf{c} \mid D\, \mathcal{M}$
PreError type	G	$::=$	$\mathbb{e}\, \tau\, \bar{B} \mid G\, \tau$
Error types	E	$::=$	$\langle G \rangle \mid E\, \tau$
Branch type	B	$::=$	$\bar{\mathcal{M}} \Rightarrow \tau$

Figure 2 Typing rules for $\mathcal{T}$

$$\frac{}{x : [\tau]; \Sigma \vdash x : \tau}\ \text{tVar} \qquad \frac{}{\varnothing; \Sigma \vdash \mathbf{c} : \mathbf{c}}\ \text{tCons}$$

$$\frac{\Gamma \oplus x :: \mathcal{M}; \Sigma \vdash t : \tau}{\Gamma; \Sigma \vdash \lambda x.t : \mathcal{M} \to \tau}\ \text{tAbs} \qquad \frac{\Gamma; \Sigma \vdash t : \tau \quad \tau \mathbin{@} \mathcal{M} \Rightarrow \sigma \quad \Delta; \Sigma \vdash s : \mathcal{M}}{\Gamma + \Delta; \Sigma \vdash ts : \sigma}\ \text{tApp}$$

$$\frac{\Gamma \oplus x :: \mathcal{M}; \Sigma \vdash t : \tau \quad \Delta; \Sigma \vdash \mathtt{fix}(x.t) : \mathcal{M}}{\Gamma + \Delta; \Sigma \vdash \mathtt{fix}(x.t) : \tau}\ \text{tFix} \qquad \frac{\Gamma; \Sigma \vdash t : \tau \quad \tau\ \langle \bar{b} \rangle\ \Delta; \Sigma, \sigma}{\Gamma + \Delta; \Sigma \vdash \mathtt{case}\ t\ \mathtt{of}\ \bar{b} : \sigma}\ \text{tCase}$$

$$\frac{\Gamma \oplus x :: \mathcal{M}; \Sigma \vdash t : \tau \qquad \Delta; \Sigma \vdash s : \mathcal{M}}{\Gamma + \Delta; \Sigma \vdash t[x \backslash s] : \tau}\ \text{tES} \qquad \frac{(\Gamma_i; \Sigma \vdash t : \tau_i)_{1 \le i \le n} \qquad (n \ge 0)}{\sum_{i=1}^{n} \Gamma_i; \Sigma \vdash t : \sum_{i=1}^{n} [\tau_i]}\ \text{tMulti}$$

$$\frac{}{\mathcal{M} \to \tau \mathbin{@} \mathcal{M} \Rightarrow \tau}\ \text{tAppFun} \qquad \frac{}{D \mathbin{@} \mathcal{M} \Rightarrow D\mathcal{M}}\ \text{tAppData} \qquad \frac{\bar{\tau} = \tau_1 \dots \tau_n}{\langle G \rangle\, \bar{\tau} \mathbin{@} [\tau_1] \Rightarrow \langle G\, \tau_1 \rangle\, \tau_2 \dots \tau_n}\ \text{tAppErr}$$

$$\frac{\mathbf{c}_j \bar{\mathcal{M}} \text{ matches } (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \qquad \Gamma \oplus \bar{x}_j :: \bar{\mathcal{M}}; \Sigma \vdash s_j : \sigma_j}{\mathbf{c}_j \bar{\mathcal{M}}\ \langle (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \rangle\ \Gamma; \Sigma, \sigma_j}\ \text{tCMatch}$$

$$\frac{\tau \text{ does not match } (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \quad \left(\Gamma_i \oplus \bar{x}_i :: \bar{\mathcal{M}}_i; \Sigma \vdash s_i : \sigma_i\right)_{i \in I}}{\tau\ \langle (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \rangle\ (\sum_{i \in I} \Gamma_i); \Sigma \cup \{\langle \mathbb{e}\, \tau\, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i \in I} \rangle\, \bar{\rho}\}, \langle \mathbb{e}\, \tau\, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i \in I} \rangle\, \bar{\rho}}\ \text{tCMismatch}$$

The type α is a type variable, $\mathcal{M} \to \tau$ is a **function type**, D is a *datatype* and E is an *error type*. A **datatype** is either a constant type $\mathbf{c}$ or an applied datatype $D\,\mathcal{M}$. Informally, $\mathbf{c}\mathcal{M}_1 \dots \mathcal{M}_n$ is the type of a constant applied to n arguments, each of which has been assigned a multiset of types. PreError types are solely introduced for building error types; error types are used for typing case expressions which will eventually become **stuck**. A case is stuck if, intuitively, it can be decided that the condition cannot match any branch. An **error type** $\langle \mathbb{e}\, \tau\, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i \in I} \rho_1 \dots \rho_j \rangle\, \rho_{j+1} \dots \rho_k$ is the type of a case expression:

1. whose condition has type τ and branches type $\bar{\mathcal{M}}_i \Rightarrow \sigma_i$;
2. which is stuck;
3. which has been applied to arguments of type $\rho_1 \dots \rho_j$; and
4. which is expecting arguments of type $\rho_{j+1} \dots \rho_k$.

We call $\mathbb{e}$ an **error type constructor**.

Letters $\Gamma, \Delta, \Theta, \dots$ range over **typing contexts**, which are functions mapping variables to multisets of types. $\Gamma(x)$ is the multiset associated to the variable x. $\operatorname{dom} \Gamma$ is the domain of Γ, namely the set of x s.t. $\Gamma(x) \neq []$.

Letters $\Sigma, \Upsilon, \dots$ range over **error logs**, sets of error types. The **sum** of typing contexts $\Gamma + \Delta$ is defined as follows: $\operatorname{dom}(\Gamma + \Delta) := \operatorname{dom} \Gamma \cup \operatorname{dom} \Delta$ and $(\Gamma + \Delta)(x) := \Gamma(x) + \Delta(x)$. The

disjoint sum of typing contexts $\Gamma \oplus \Delta$ is defined as $\Gamma + \Delta$ provided $\mathrm{dom}\,\Gamma \cap \mathrm{dom}\,\Delta = \varnothing$. We write $\Gamma, x : \mathcal{M}$ for $\Gamma + \{x : \mathcal{M}\}$ and $\Gamma, x :: \mathcal{M}$ for $\Gamma \oplus \{x : \mathcal{M}\}$. Also, we write $\bar{x} : \bar{\mathcal{M}}$ for $((x_i)_{i \in I} : (\mathcal{M}_i)_{i \in I}) := \sum_{i \in I}(x_i : \mathcal{M}_i)$ and similarly for $\bar{x} :: \bar{\mathcal{M}}$.

Definition 16. *The typing system $\mathcal{T}$ is defined by means of the typing rules of Fig. 2. These rules introduce four, mutually recursive,* **typing judgements***:*

1. **Typing** *($\Gamma; \Sigma \vdash t : \tau$)*
   Term t has type τ under context Γ and error log Σ.

2. **Multi-typing** *($\Gamma; \Sigma \vdash t : \mathcal{M}$)*
   Term t has the types in $\mathcal{M}$ under context Γ and error log Σ.

3. **Application** *($\tau \mathbin{@} \mathcal{M} \Rightarrow \sigma$)*
   A term of type τ may be applied to an argument that has all the types in $\mathcal{M}$, resulting in a term of type σ.

4. **Matching** *($\tau \, \langle \bar{b} \rangle \, \Gamma; \Sigma, \sigma$)*
   Type τ might be the condition of a case with branches $\bar{b}$, which will result in a term of type σ, provided certain hypotheses Γ and error logs Σ, or else fail.

We write $\pi, \xi, \ldots$ for typing derivations and $\pi(\Gamma; \Sigma \vdash t : \tau)$ if π is a typing derivation of the judgement $\Gamma; \Sigma \vdash t : \tau$. We comment on the salient typing rules. The axioms are *linear* w.r.t. the typing context in that they require the typing context to be empty but for the type assigned to x in tVar and the typing context to be empty in tCons. The error context, however, is said to be *intuitionistic* in that it may hold any number of error types. Rule tApp caters for typing applications of terms of functional type, data structures and error terms, to arguments by means of the *application judgement* $\tau \mathbin{@} \mathcal{M} \Rightarrow \sigma$. Indeed, τ may be of the form $\mathcal{M} \to \sigma$ (*cf.* tAppFun), or a datatype D in which case σ is $D\mathcal{M}$ (*cf.* tAppData), or an error type $\langle G \rangle \, \tau_1 \ldots \tau_n$ in which case $\mathcal{M}$ must be a singleton $[\tau_1]$ and σ of the $\langle G \, \tau_1 \rangle \, \tau_2 \ldots \tau_n$ (*cf.* tAppErr). The restriction to a singleton type in the last case is due to the fact that all one wants to do is enforce that the arguments of a stuck case be typable. Note also that typing contexts are *multiplicative* whereas error logs are *additive*. The tFix splits its resources so that they are dealt out to be used for one unfolding (Γ) and the rest of the unfoldings (Δ). The tCase rule relies on the *matching judgement* $\tau \, \langle \bar{b} \rangle \, \Delta; \Sigma, \sigma$. The latter checks whether the type of the condition τ *matches* the list of branches. A **type** τ **matches** with a branch $\mathbf{c}\bar{x} \Rightarrow s$ if $\tau = \mathbf{c}\bar{\mathcal{M}}$ with $|\bar{\mathcal{M}}| = |\bar{x}|$. A type matches with a list of branches if it matches with at least one branch. Returning to our description of tCase, if τ matches with a branch, then that branch is typed (*cf.* tCMatch). However, if τ does not match any branch (*cf.* tCMismatch), then *all* branches have to be accounted for by the type system. Moreover, the type of the case expression will be an error type of the form $\langle \mathbb{e} \, \tau \, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i \in I} \rangle \, \bar{\rho}$, which is recorded in the error log. Note that $\bar{\rho} = \rho_1, \ldots, \rho_k$ are the types of the arguments to which the stuck case expression will be allowed to be applied to. Finally, tMulti allows a term to be typed with a multiset type. In this rule, if $n = 0$, then $\sum_{i=1}^{n} [\tau_i]$ denotes the empty multiset $[]$.

Remark 17. *$\mathcal{T}$ does not enjoy unique typing. For example, it is possible to assign many types the expression* **cons zero** (**cons zero nil**)*, namely* **cons** $[]\,[]$ *or* **cons** $[$**zero**$]\,[]$*, or* **cons** $[$**zero**, **zero**$]\,[]$.

Figure 3 Example derivation

(a)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{n:[\mathbf{z}];\varnothing\vdash n:\mathbf{z}}\ \textsc{tVar} \qquad \dfrac{\pi_1}{\mathbf{z}\ \langle\bar{b}\rangle\ \varnothing;\varnothing,\mathbf{s}\,[\mathbf{z}]}}{n:[\mathbf{z}];\varnothing\vdash \texttt{case}\ n\ \texttt{of}\ \bar{b}:\mathbf{s}\,[\mathbf{z}]}\ \textsc{tCase}}{\varnothing;\varnothing\vdash\lambda n.\texttt{case}\ n\ \texttt{of}\ \bar{b}:[\mathbf{z}]\to\mathbf{s}\,[\mathbf{z}]}\ \textsc{tAbs} \qquad \dfrac{}{\varnothing;\varnothing\vdash t:[\,]}\ \textsc{TMulti}}{\varnothing;\varnothing\vdash t:[\mathbf{z}]\to\mathbf{s}\,[\mathbf{z}]}\ \textsc{tFix}$$

(b)

$$\dfrac{\dfrac{\dfrac{}{\varnothing;\varnothing\vdash\mathbf{s}:\mathbf{s}}\ \textsc{tCons} \qquad \dfrac{}{\mathbf{s}\ @\ [\mathbf{z}]\Rightarrow\mathbf{s}\,[\mathbf{z}]}\ \textsc{tAppData} \qquad \dfrac{\dfrac{}{\varnothing;\varnothing\vdash\mathbf{z}:\mathbf{z}}\ \textsc{tCons}}{\varnothing;\varnothing\vdash\mathbf{z}:[\mathbf{z}]}\ \textsc{tMulti}}{\varnothing;\varnothing\vdash\mathbf{s}\,\mathbf{z}:\mathbf{s}\,[\mathbf{z}]}\ \textsc{tApp}}{\mathbf{z}\ \langle(\mathbf{z}\Rightarrow\mathbf{s}\,\mathbf{z};\mathbf{s}\,n\Rightarrow\mathbf{s}\,n*f\,n)\rangle\ \varnothing;\varnothing,\mathbf{s}\,[\mathbf{z}]}\ \textsc{tCM}$$

(c)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{n:[\mathbf{s}\,[\mathbf{z},\mathbf{z}]];\varnothing\vdash n:\mathbf{s}\,[\mathbf{z},\mathbf{z}]}\ \textsc{TVar} \qquad \dfrac{\xi_1}{\mathbf{s}\,[\mathbf{z},\mathbf{z}]\ \langle\bar{b}\rangle\ \{f:[[\mathbf{z}]\to\mathbf{s}\,[\mathbf{z}]]\};\varnothing,\mathbf{s}\,[\mathbf{z}]}}{f:[[\mathbf{z}]\to\mathbf{s}\,[\mathbf{z}]],n:[\mathbf{s}\,[\mathbf{z},\mathbf{z}]];\varnothing\vdash\texttt{case}\ n\ \texttt{of}\ \bar{b}:\mathbf{s}\,[\mathbf{z}]}\ \textsc{tCase}}{f:[[\mathbf{z}]\to\mathbf{s}\,[\mathbf{z}]];\varnothing\vdash\lambda n.\texttt{case}\ n\ \texttt{of}\ \bar{b}:[\mathbf{s}\,[\mathbf{z},\mathbf{z}]]\to\mathbf{s}\,[\mathbf{z}]}\ \textsc{tAbs} \qquad \dfrac{\pi}{\varnothing\vdash t:[[\mathbf{z}}}{\varnothing\vdash t:[\mathbf{s}\,[\mathbf{z},\mathbf{z}]]\to\mathbf{s}\,[\mathbf{z}]}$$

(d)

$$\dfrac{\dfrac{\dfrac{\xi_2}{n:[\mathbf{z}];\varnothing\vdash\mathbf{s}\,n:\mathbf{s}\,[\mathbf{z}]} \qquad \dfrac{f:[[\mathbf{z}]\to\mathbf{s}\,[\mathbf{z}]];\varnothing\vdash f:[\mathbf{z}]\to\mathbf{s}\,[\mathbf{z}] \quad [\mathbf{z}]\to\mathbf{s}\,[\mathbf{z}]\ @\ [\mathbf{z}]\Rightarrow\mathbf{s}\,[\mathbf{z}] \quad n:[\mathbf{z}];\varnothing\vdash n:[\mathbf{z}]}{n:[\mathbf{z}],f:[[\mathbf{z}]\to\mathbf{s}\,[\mathbf{z}]];\varnothing\vdash f\,n:\mathbf{s}\,[\mathbf{z}]}\ \textsc{T}}{f:[[\mathbf{z}]\to\mathbf{s}\,[\mathbf{z}]],n:[\mathbf{z},\mathbf{z}];\varnothing\vdash\mathbf{s}\,n*f\,n:\mathbf{s}\,[\mathbf{z}]}}{\mathbf{s}\,[\mathbf{z},\mathbf{z}]\ \langle\bar{b}\rangle\ \{f:[[\mathbf{z}]\to\mathbf{s}\,[\mathbf{z}]]\};\varnothing,\mathbf{s}\,[\mathbf{z}]}$$

3.1 An Example

Let t be the term $\mathtt{fix}(f.\lambda n.\mathtt{case}\ n\ \mathtt{of}\ \mathbf{z} \Rightarrow \mathbf{s}\,\mathbf{z}; \mathbf{s}\,n \Rightarrow \mathbf{s}\,n * f\,n)$ representing the factorial function. We exhibit type derivations for the judgements:

1. $\varnothing; \varnothing \vdash t : [\mathbf{z}] \to \mathbf{s}\,[\mathbf{z}]$; and
2. $\varnothing; \varnothing \vdash t : [\mathbf{s}\,[\mathbf{z}, \mathbf{z}]] \to \mathbf{s}\,[\mathbf{z}]$.

We use $\bar{b}$ to denote the branches $\mathbf{z} \Rightarrow \mathbf{s}\,\mathbf{z}; \mathbf{s}\,n \Rightarrow \mathbf{s}\,n * f\,n$. The derivation π for the first judgement is in Fig. 3(a). Note the absence of f in the typing context of the judgement

$$\varnothing; \varnothing \vdash \lambda n.\mathtt{case}\ n\ \mathtt{of}\ \mathbf{z} \Rightarrow \mathbf{s}\,\mathbf{z}; \mathbf{s}\,n \Rightarrow \mathbf{s}\,n * f\,n : [\mathbf{z}] \to \mathbf{s}\,[\mathbf{z}]$$

Since the type of n is $[\mathbf{z}]$ the branch with the recursive call will not be used and hence is not typed. The missing subderivation of π called π_1, of the judgement $\mathbf{z}\ \langle\bar{b}\rangle\ \varnothing, \mathbf{s}\,[\mathbf{z}]$, is given in Fig 3(b). Since $\mathbf{z}$ matches with $(\mathbf{z} \Rightarrow \mathbf{s}\,\mathbf{z})$, we only type this branch.

A derivation ξ of $\varnothing \vdash t : [\mathbf{s}\,[\mathbf{z}, \mathbf{z}]] \to \mathbf{s}\,[\mathbf{z}]$ is in Fig. 3(c). We use the following typing rule for the product:

$$\frac{\Gamma; \Sigma \vdash t : \mathbf{s}^n\,\mathbf{z} \qquad \Delta; \Sigma \vdash s : \mathbf{s}^m\,\mathbf{z}}{\Gamma + \Delta; \Sigma \vdash t * s : \mathbf{s}^{n*m}\,\mathbf{z}}\ \textsc{tProd}$$

The derivation ξ_1 of $\mathbf{s}\,[\mathbf{z}, \mathbf{z}]\ \langle\bar{b}\rangle\ \{f : [[\mathbf{z}] \to \mathbf{s}\,[\mathbf{z}]]\}; \varnothing, \mathbf{s}\,[\mathbf{z}]$ is given in Fig. 3(d). The missing subderivation ξ_2 may be completed without trouble.

3.2 Metatheory

Lemma 18 (Generation Lemma for $\mathcal{T}$)**.** *Let π be a derivation in $\mathcal{T}$.*

1. $\pi(\Gamma; \Sigma \vdash x : \tau)$ *implies* $\Gamma = x : [\tau]$.
2. $\pi(\Gamma; \Sigma \vdash \lambda x.t : \tau)$ *implies* $\exists \sigma$ *and* $\mathcal{M}$ *s.t.* $\tau = \mathcal{M} \to \sigma$ *and* $\Gamma \oplus x :: \mathcal{M}; \Sigma \vdash t : \sigma$.
3. $\pi(\Gamma; \Sigma \vdash \mathbf{c} : \mathbf{c})$ *implies* $\Gamma = \varnothing$.
4. $\pi(\Gamma; \Sigma \vdash ts : \tau)$ *implies* $\exists \Gamma_1, \Gamma_2, \sigma, \mathcal{M}$ *s.t.* $\Gamma = \Gamma_1 + \Gamma_2$ *and* $\Gamma_1; \Sigma \vdash t : \sigma$, $\sigma\ @\ \mathcal{M} \Rightarrow \tau$ *and* $\Gamma_2; \Sigma \vdash s : \mathcal{M}$
5. $\pi(\Gamma; \Sigma \vdash \mathtt{fix}(x.t) : \tau)$ *implies* $\exists \Gamma_1, \Gamma_2, \mathcal{M}$ *s.t.* $\Gamma = \Gamma_1 + \Gamma_2$, *and* $\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash t : \tau$ *and* $\Gamma_2; \Sigma \vdash \mathtt{fix}(x.t) : \mathcal{M}$
6. $\pi(\Gamma; \Sigma \vdash \mathtt{case}\ t\ \mathtt{of}\ \bar{b} : \tau)$ *implies* $\exists \Gamma_1, \Gamma_2$ *s.t.* $\Gamma = \Gamma_1 + \Gamma_2$, *and* $\Gamma_1; \Sigma \vdash t : \sigma$ *and* $\sigma\ \langle\bar{b}\rangle\ \Gamma_2; \Sigma, \tau$.
7. $\pi(\Gamma; \Sigma \vdash t[x \backslash s] : \tau)$ *implies* $\exists \Gamma_1, \Gamma_2, \mathcal{M}$ *s.t.* $\Gamma = \Gamma_1 + \Gamma_2$, *and* $\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash t : \tau$ *and* $\Gamma_2; \Sigma \vdash s : \mathcal{M}$ *are derivable.*
8. $\pi(\Gamma; \Sigma \vdash t : \mathcal{M})$ *implies* $\exists n, \Gamma_i, \tau_i$, *for* $i \in 1..n$, *s.t.* $\Gamma = \sum_{i=1}^{n} \Gamma_i$ *and* $\mathcal{M} = \sum_{i=1}^{n} [\tau_i]$ *and* $\pi_i(\Gamma_i; \Sigma \vdash t : \tau_i)$, *for* $i \in 1..n$.

Proof. By induction on π. □

Lemma 19. *If* $\pi(\Gamma; \Sigma \vdash t : \tau)$, *then* $\mathrm{dom}\,\Gamma \subseteq \mathsf{fv}(t)$.

Proof. By induction on π. $\square$

Lemma 20 (Type of Answers). *If $\pi(\Gamma; \Sigma \vdash t : \tau)$, then*

- $t = \mathtt{A}[\mathbf{c}]\mathtt{L}$ *implies* $\tau = \mathbf{c}\,\mathcal{M}_1 \ldots \mathcal{M}_n$ *where* $n = |\mathtt{A}|$.
- $t = (\lambda x.s)\mathtt{L}$ *implies* $\tau = \mathcal{M} \to \sigma$.

Proof. By induction on π. $\square$

Lemma 21 (Reverse substitution). *If $\Gamma; \Sigma \vdash t\{x := s\} : \tau$ then there exist contexts Γ_1, Γ_2, and a multiset of types $\mathcal{M}$ such that:*

1. $\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash t : \tau$;
2. $\Gamma_2; \Sigma \vdash s : \mathcal{M}$; *and*
3. $\Gamma = \Gamma_1 + \Gamma_2$.

Proof. By induction on the derivation of $\Gamma; \Sigma \vdash t\{x := s\} : \tau$. Note that if $t = x$, then we set $\Gamma_1 := \varnothing$, $\mathcal{M} := [\tau]$, and $\Gamma_2 := \Gamma$, and the result holds trivially. So we henceforth assume that $t \neq x$. A word on notation in the proof below: we write $\Gamma_1\Gamma_2$ as shorthand for $\Gamma_1 + \Gamma_2$.

- tVar. $t\{x := s\} = y$ and the derivation ends in:

$$\frac{}{y : [\tau]; \Sigma \vdash y : \tau}\ \text{tVar}$$

  Given the above mentioned comment, $t = y$. We set $\Gamma_1 := y : [\tau]$, $\mathcal{M} := []$.

- tAbs. $t\{x := s\} = \lambda y.u$ and $\tau = \mathcal{N} \to \sigma$ and the derivation ends in:

$$\frac{\Gamma \oplus y :: \mathcal{N}; \Sigma \vdash u : \sigma}{\Gamma; \Sigma \vdash \lambda y.u : \mathcal{N} \to \sigma}\ \text{tAbs}$$

  Then $t = \lambda y.r$ and $u = r\{x := s\}$. By the *i.h.* there exist Γ'_1, Γ'_2, and $\mathcal{M}'$ s.t.

$$\Gamma'_1 \oplus x :: \mathcal{M}'; \Sigma \vdash r : \sigma \qquad \Gamma'_2; \Sigma \vdash s : \mathcal{M}' \qquad \Gamma \oplus y :: \mathcal{N} = \Gamma'_1 + \Gamma'_2 (= \Gamma''_1 \oplus y :: \mathcal{N} + \Gamma'_2)$$

  From $\Gamma''_1 \oplus y :: \mathcal{N} \oplus x :: \mathcal{M}'; \Sigma \vdash r : \sigma$ we deduce $\Gamma''_1 \oplus x :: \mathcal{M}'; \Sigma \vdash \lambda y.r : \mathcal{N} \to \sigma$. Thus we set $\Gamma_1 := \Gamma''_1$, $\Gamma_2 := \Gamma'_2$, and conclude.

- tCons. $t\{x := s\} = \mathbf{c}$ and the derivation ends in:

$$\frac{}{\varnothing; \Sigma \vdash \mathbf{c} : \mathbf{c}}\ \text{tCons}$$

  Then $t = \mathbf{c}$ and we set $\Gamma_1 := \varnothing$, $\mathcal{M} := []$, and $\Gamma_2 := \varnothing$, and the result holds trivially.

- TAPP. $t\{x := s\} = ur$ and the derivation ends in:

$$\frac{\Gamma_1; \Sigma \vdash u : \sigma \qquad \sigma \mathbin{@} \mathcal{P} \Rightarrow \tau \qquad \Gamma_2; \Sigma \vdash r : \mathcal{P}}{\Gamma_1 + \Gamma_2; \Sigma \vdash ur : \tau}\ \textsc{tApp}$$

where, $\Gamma = \Gamma_1 + \Gamma_2$. Given the above mentioned observation, $t = t_1\, t_2$, for some terms t_1 and t_2, and $u = t_1\{x := s\}$ and $r = t_2\{x := s\}$. Thus we have:

$$\frac{\Gamma_1; \Sigma \vdash t_1\{x := s\} : \sigma \qquad \sigma \mathbin{@} \mathcal{P} \Rightarrow \tau \qquad \Gamma_2; \Sigma \vdash t_2\{x := s\} : \mathcal{P}}{\Gamma_1 + \Gamma_2; \Sigma \vdash (t_1\, t_2)\{x := s\} : \tau}\ \textsc{tApp}$$

From the *i.h.* on $\Gamma_1; \Sigma \vdash t_1\{x := s\} : \sigma$ we know there exist Γ_{11}, Γ_{12}, and $\mathcal{N}_1$ s.t.

$$\Gamma_{11} \oplus x :: \mathcal{N}_1; \Sigma \vdash t_1 : \sigma \quad \Gamma_{12}; \Sigma \vdash s : \mathcal{M}_1 \quad \Gamma_1 = \Gamma_{11} + \Gamma_{12}$$

Let $\mathcal{P} = [\rho_1, \ldots \rho_k]$. Then $\Gamma_2 = \sum_{i=1}^{k} \Gamma_{2i}$. From the *i.h.* on each $\Gamma_{2i}; \Sigma \vdash t_2\{x := s\} : \rho_i$, for $i \in 1..k$, we know there exists Γ_{2i1}, Γ_{2i2}, and $\mathcal{N}_{2i}$ s.t.

$$\Gamma_{2i1} \oplus x :: \mathcal{N}_{2i}; \Sigma \vdash t_2 : \rho_i \quad \Gamma_{2i2}; \Sigma \vdash s : \mathcal{N}_{2i} \quad \Gamma_{2i} = \Gamma_{2i1} + \Gamma_{2i2}$$

We thus have $\sum_{i=1}^{k} \Gamma_{2i1} \oplus \sum_{i=1}^{k} \mathcal{M}_{2i}; \Sigma \vdash t_2 : \mathcal{P}$ and therefore, using TAPP, also $(\Gamma_{11} \oplus x :: \mathcal{M}_1) + (\sum_{i=1}^{k} \Gamma_{2i1} \oplus \sum_{i=1}^{k} \mathcal{M}_{2i}); \Sigma \vdash t_1\, t_2 : \sigma$. We set $\mathcal{M} := \mathcal{M}_1 + \sum_{i=1}^{k} \mathcal{M}_{2i}$ and $\Gamma_1 := \Gamma_{11} + \sum_{i=1}^{k} \Gamma_{2i1}$. This proves the first item.

For the second item we set $\Gamma_2 := \Gamma_{12} + \sum_{i=1}^{k} \Gamma_{2i2}$.

We are left to verify that $\Gamma = \Gamma_1 + \Gamma_2 = \Gamma_{11} + \Gamma_{12} + \sum_{i=1}^{k} \Gamma_{2i}$. We reason as follows:

$$\begin{array}{ll} & \Gamma_{11} + \Gamma_{12} + \sum_{i=1}^{k} \Gamma_{2i} \\ = & \Gamma_{11} + \Gamma_{12} + \sum_{i=1}^{k} (\Gamma_{2i1} + \Gamma_{2i2}) \\ = & \Gamma_{11} + \sum_{i=1}^{k} \Gamma_{2i1} + \Gamma_{12} + \sum_{i=1}^{k} \Gamma_{2i2} \\ = & \Gamma_1 + \Gamma_2 \end{array}$$

- TFIX. $t\{x := s\} = \mathtt{fix}(y.u)$ and the derivation ends in:

$$\frac{\Gamma_1 \oplus y :: \mathcal{N}; \Sigma \vdash u : \tau \qquad \Gamma_2; \Sigma \vdash \mathtt{fix}(y.u) : \mathcal{N}}{\Gamma_1 + \Gamma_2; \Sigma \vdash \mathtt{fix}(y.u) : \tau}\ \textsc{tFix}$$

and $\Gamma = \Gamma_1 + \Gamma_2$. Then $t = \mathtt{fix}(y.t_1)$, $u = t_1\{x := s\}$ and by the *i.h.* on $\Gamma_1 \oplus y :: \mathcal{N}; \Sigma \vdash u : \tau$, there exists Γ_{11}, Γ_{12} and $\mathcal{M}_1$ s.t.

$$\Gamma_{11} \oplus y :: \mathcal{N} \oplus x : \mathcal{M}_1; \Sigma \vdash t_1 : \tau \quad \Gamma_{12}; \Sigma \vdash s : \mathcal{M}_1 \quad \Gamma_1 = \Gamma_{11} \oplus y :: \mathcal{N} + \Gamma_{12} \tag{3}$$

Similarly, by the *i.h.* on $\Gamma_2^i; \Sigma \vdash \mathtt{fix}(y.u) : \sigma_i$, where $\mathcal{N} = [\sigma_1, \ldots, \sigma_k]$ and $\Gamma_2 = \sum_{i=1..k} \Gamma_2^i$, there exists $\Gamma_{21}^i, \Gamma_{22}^i$ and $\mathcal{M}_2^i$ s.t.

$$\Gamma_{21}^i \oplus x : \mathcal{M}_2^i; \Sigma \vdash \mathtt{fix}(y.t_1) : \sigma_i \quad \Gamma_{22}^i; \Sigma \vdash s : \mathcal{M}_2^i \quad \Gamma_2^i = \Gamma_{21}^i \oplus y :: \mathcal{M}_2^i + \Gamma_{22}^i \tag{4}$$

Let $\Gamma_{21} = \sum_{i=1..k} \Gamma_{21}^i$ and $\Gamma_{22} = \sum_{i=1..k} \Gamma_{22}^i$ and $\mathcal{M}_2 = \sum_{i=1..k} \mathcal{M}_2^i$. Then $\Gamma_{21} \oplus x : \mathcal{M}_2; \Sigma \vdash \mathtt{fix}(y.t_1) : \mathcal{N}$. Thus we can derive:

$$\frac{\Gamma_{11} \oplus y :: \mathcal{N} \oplus x : \mathcal{M}_1; \Sigma \vdash t_1 : \tau \qquad \Gamma_{21} \oplus x : \mathcal{M}_2; \Sigma \vdash \mathtt{fix}(y.t_1) : \mathcal{N}}{\Gamma_{11} + \Gamma_{21} \oplus x : \mathcal{M}_1 + \mathcal{M}_2; \Sigma \vdash \mathtt{fix}(y.u) : \tau} \text{TFIX}$$

Notice also that:

$$\Gamma_{12}\Gamma_{22}; \Sigma \vdash s : \mathcal{M}_1 \cup \mathcal{M}_2$$

Thus we set $\mathcal{M} := \mathcal{M}_1 + \mathcal{M}_2$, $\Gamma_1 := \Gamma_{11}\Gamma_{21}$ and $\Gamma_2 := \Gamma_{21}\Gamma_{22}$.

- TCASE. $t\{x := s\} = \mathtt{case}\ u\ \mathtt{of}\ \bar{b}$ and the derivation ends in:

$$\frac{\Gamma_1; \Sigma \vdash u : \sigma \qquad \sigma\ \langle \bar{b} \rangle\ \Gamma_2; \Sigma, \tau}{\Gamma_1 + \Gamma_2; \Sigma \vdash \mathtt{case}\ u\ \mathtt{of}\ \bar{b} : \tau} \text{TCASE}$$

and $\Gamma = \Gamma_1 + \Gamma_2$. First let us assume that $\sigma = \mathbf{c}_j \bar{\mathcal{M}}$ and $\mathbf{c}_j \bar{\mathcal{M}}$ matches with $\bar{b} = (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I}$. Then $\Gamma_2 \oplus \bar{x}_j :: \bar{\mathcal{M}}; \Sigma \vdash s_j : \sigma_j$ and $\tau = \sigma_j$. In this case we use the *i.h.* on $\Gamma_1; \Sigma \vdash u : \sigma$ and $\Gamma_2 \oplus \bar{x}_j :: \bar{\mathcal{M}}; \Sigma\ \vdash s_j : \sigma_j$.

If σ does not match with $\bar{b} = (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I}$, then we know that $(\Gamma_{2i} \oplus \bar{x}_i :: \bar{\mathcal{M}}_i; \Sigma' \vdash s_i : \sigma_i)_{i \in I}$ and $\Sigma = \Sigma' \cup \{\rho\}$ and $\rho := \langle \mathtt{e}\, \tau\, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i \in I} \rangle\, \bar{\rho}$ and also $\Gamma_2 = \sum_{i \in I} \Gamma_{2i}$ and $\tau = \rho$. Here we use the *i.h.* on $\Gamma_1; \Sigma \vdash u : \sigma$ and on each of $(\Gamma_{2i} \oplus \bar{x}_i :: \bar{\mathcal{M}}_i; \Sigma' \vdash s_i : \sigma_i)_{i \in I}$.

$\square$

4 Towards Completeness of The Strategy

We next outline the proof method [BBBK17] that we use to prove that the strong call-by-need strategy (to be introduced in Sec. 5) is correctly behaved w.r.t. reduction in $\lambda_{\mathtt{e}}$.

- **Step 1 (Sec. 4.1)**. Weakly normalising terms of $\lambda_{\mathtt{e}}$ are typable in the non-idempotent intersection type system $\mathcal{T}$.
- **Step 2 (Sec. 4.2).** Typable terms in $\mathcal{T}$ are weakly normalising in the theory of sharing $\lambda_{\mathtt{sh}}$.
- **Step 3 (Sec. 6).** Factorization of the reduction sequence in $\lambda_{\mathtt{sh}}$ obtained in Step 2 into an *external* part followed by an *internal* part, by means of a *standardisation theorem*. The former part corresponds to the strong call-by-need strategy and the latter shown to be superfluous. The end term of the external part is shown to be identical modulo unsharing (or unfolding of explicit substitutions) to the original normal form from Step 1.

A corollary is that the strong call-by-need strategy is complete w.r.t reduction in the Extended Lambda Calculus: if t reduces to a normal form s in $\lambda_{\mathtt{e}}$, then the strategy computes a normal form u such that s is the unsharing of u.

4.1 Definable Terms in λ_{e} are Typable (Step 1)

This section addresses Step 1 of the diagram below, where $t \in \lambda_{\mathsf{e}}$:

$$t \in \mathsf{WN}(\lambda_{\mathsf{e}}) \overset{\textit{Step 1}}{\overbrace{\Rightarrow}} t \in \mathsf{Typable}(\mathcal{T}) \overset{\textit{Step 2}}{\overbrace{\Rightarrow}} t \in \mathsf{WN}(\lambda_{\mathtt{sh}})$$

As discussed in the Introduction, we don't want t to *just* be typable in $\mathcal{T}$ but to be typable with some additional *constraints* so that Step 2 holds too. For example, we mentioned that $x\,\Omega$ is typable by setting x to have type $[] \to \alpha$, for α a type variable; however, the term is not normalising in $\lambda_{\mathtt{sh}}$. We must require that the typing judgement $\Gamma; \Sigma \vdash t : \tau$ be such that $[] \notin \mathcal{N}(\Gamma)$ and $[] \notin \mathcal{P}(\tau)$ [BKV17]. Here $\mathcal{N}(\Gamma)$ and $\mathcal{P}(\tau)$ refer to the usual notions of negative and positive occurrences of types in τ (*cf.* Fig. 4). In the presence of constants and case expressions, this constraint does not suffice. We introduce an extended set of constraints below that determines what we call **good judgements** (*cf.* Def. 22). We revisit below some examples from the introduction to motivate them.

Consider the term $\mathtt{case}\,\mathbf{c}\,\mathtt{of}\,(\mathbf{d} \Rightarrow \mathbf{d})\,\Omega$. It is typable with, for example, type $\langle \mathbb{e}\,\mathbf{c}\,([\bar{\mathbf{d}}] \Rightarrow \mathbf{d})_{i\in I}[]\rangle$. Notice how the type of the blocked case includes occurrences of the types of arguments to which it applies (in this case the empty multiset type). This will allow us to extend the above mentioned constraint to blocked case expressions.

Consider now the term $\mathtt{case}\;x\;\mathtt{of}\;(\mathbf{c} \Rightarrow \mathbf{d}) \cdot (\mathbf{e} \Rightarrow \Omega)$. This term is typable with type $\mathbf{d}$, if $x : [\mathbf{c}]$, however, it is not weakly normalizing in $\lambda_{\mathtt{sh}}$. This motivates the new constraints $\mathbf{c} \notin \mathcal{P}(\Gamma)$, $\mathbf{c} \notin \mathcal{N}(\Sigma)$, and $\mathbf{c} \notin \mathcal{N}(\tau)$ in Def. 22. In particular, in a term such as $\mathtt{case}\;x\;\mathtt{of}\;(\mathbf{c} \Rightarrow \mathbf{d}) \cdot (\mathbf{e} \Rightarrow \mathbf{d})$ which is in normal form, we will type it by assigning x an appropriate error type.

Finally, note that in pure lambda terms all terms in weak head normal form have a variable at the head. Since the types of all variables are in the typing context Γ, we can place restrictions on their type through Γ. For example, in the above mentioned term $x\,\Omega$ one may require that $[] \notin \Gamma(x)$ to force the type system to account for Ω. However, now we may have term in weak head normal form headed by blocked case expressions. In order to have access to their types so that we may place restrictions on them, we have to record them. This is the role played by the error logs and motivates the third, and final, item of our notion of good judgements, namely that all occurrences of error types be accounted for in the error log.

We will make use of the following notions on error types. The first one, $\overrightarrow{E}$, establishes a canonical notation for error type E as defined below. The second is $E_1 \simeq E_2$ that holds if $\overrightarrow{E_1} = \overrightarrow{E_2}$.

$$\begin{array}{rcl}
\overrightarrow{\langle \mathbb{e}\,\tau\,\bar{B}\rangle} & := & \langle \mathbb{e}\,\tau\,\bar{B}\rangle \\
\overrightarrow{\langle G\,\tau\rangle} & := & \overrightarrow{\langle G\rangle}\,\tau \\
\overrightarrow{E\,\tau} & := & \overrightarrow{E}\,\tau
\end{array}$$

Definition 22 (Good types and typing judgements)**.** *The set of positive (resp. negative) types occurring in τ, denoted $\mathcal{P}(\tau)$ (resp. $\mathcal{N}(\tau)$), is defined in Fig 4. A* **type τ is good** *if $\mathbf{c} \notin \mathcal{P}(\tau)$ and $[] \notin \mathcal{N}(\tau)$. We say $\mathcal{M}$ is good if each $\tau \in \mathcal{M}$ is good. A* **typing context Γ is good** *if $\Gamma = \Gamma_g\Gamma_e$ and $\forall x \in \operatorname{dom}\Gamma_g$, $\Gamma_g(x)$ is good and $\forall x \in \operatorname{dom}\Gamma_e$, $\Gamma_e(x)$ is an error type. A* **typing judgement** $\Gamma; \Sigma \vdash t : \tau$ **is good** *if*

- Γ *is good;*
- $[] \notin \mathcal{P}(\Sigma)$ *and* $[] \notin \mathcal{P}(\tau)$*;*

Figure 4 Positive and negative types

$$
\begin{array}{rcl}
\mathcal{P}(\alpha) & := & \{\alpha\} \\
\mathcal{P}(\mathcal{M} \to \tau) & := & \mathcal{N}(\mathcal{M}) \cup \mathcal{P}(\tau) \cup \{\mathcal{M} \to \tau\} \\
\mathcal{P}(\mathbf{c}) & := & \{\mathbf{c}\} \\
\mathcal{P}(D\,\mathcal{M}) & := & \mathcal{P}(D) \cup \mathcal{P}(\mathcal{M}) \cup \{D\,\mathcal{M}\} \\
\mathcal{P}(E\,\tau) & := & \mathcal{P}(E) \cup \mathcal{P}(\tau) \cup \{E\,\tau\} \\
\mathcal{P}(\langle G\rangle) & := & \mathcal{P}(G) \cup \{G\} \\
\mathcal{P}(G\,\tau) & := & \mathcal{P}(G) \cup \mathcal{P}(\tau) \cup \{G\,\tau\} \\
\mathcal{P}(\mathbb{e}\,\tau\,\bar{B}) & := & \mathcal{P}(\tau) \cup \mathcal{P}(\bar{B}) \cup \{\mathbb{e}\,\tau\,\bar{B}\} \\
\mathcal{P}(\mathcal{M}_1, \ldots, \mathcal{M}_n \Rightarrow \tau) & := & \bigcup_{i\in 1..n} \mathcal{N}(\mathcal{M}_i) \cup \mathcal{P}(\tau) \cup \{\bar{\mathcal{M}} \Rightarrow \tau\} \\
\mathcal{P}(\mathcal{M}) & := & \bigcup_{\tau\in\mathcal{M}} \mathcal{P}(\tau) \cup \{\mathcal{M}\} \\
\mathcal{P}(\Gamma; \Sigma \vdash \tau) & := & \mathcal{N}(\Gamma) \cup \mathcal{P}(\Sigma) \cup \mathcal{P}(\tau) \\
\mathcal{P}(\Gamma) & := & \bigcup \mathcal{P}(\Gamma(x)),\ \text{for all } x \in \operatorname{dom}\Gamma
\end{array}
\qquad
\begin{array}{rcl}
\mathcal{N}(\alpha) & := & \varnothing \\
\mathcal{N}(\mathcal{M} \to \tau) & := & \mathcal{P}(\mathcal{M}) \cup \mathcal{N}(\tau) \\
\mathcal{N}(\mathbf{c}) & := & \varnothing \\
\mathcal{N}(D\,\mathcal{M}) & := & \mathcal{N}(D) \cup \mathcal{N}(\mathcal{M}) \\
\mathcal{N}(E\,\tau) & := & \mathcal{N}(E) \cup \mathcal{N}(\tau) \\
\mathcal{N}(\langle G\rangle) & := & \mathcal{N}(G) \\
\mathcal{N}(G\,\tau) & := & \mathcal{N}(G) \cup \mathcal{N}(\tau) \\
\mathcal{N}(\mathbb{e}\,\tau\,\bar{B}) & := & \mathcal{N}(\tau)\ \cup \mathcal{N}(\bar{B}) \\
\mathcal{N}(\mathcal{M}_1, \ldots, \mathcal{M}_n \Rightarrow \tau) & := & \bigcup_{i\in 1..n} \mathcal{P}(\mathcal{M}_i) \cup \mathcal{N}(\tau) \\
\mathcal{N}(\mathcal{M}) & := & \bigcup_{\tau\in\mathcal{M}} \mathcal{N}(\tau) \\
\mathcal{N}(\Gamma; \Sigma \vdash \tau) & := & \mathcal{P}(\Gamma) \cup \mathcal{N}(\Sigma) \cup \mathcal{N}(\tau) \\
\mathcal{N}(\Gamma) & := & \bigcup \mathcal{N}(\Gamma(x)),\ \text{for all } x \in \operatorname{dom}\Gamma
\end{array}
$$

$$
\begin{array}{rcl}
\mathsf{covered}_\Sigma(\Gamma) & := & \bigwedge_{x\in\operatorname{dom}\Gamma} \mathsf{covered}_\Sigma(\Gamma(x)) \\
\\
\mathsf{covered}_\Sigma(\alpha)_T & := & \mathit{true} \\
\mathsf{covered}_\Sigma(\mathcal{M} \to \tau)_T & := & \mathsf{covered}_\Sigma(\mathcal{M})_T \wedge \mathsf{covered}_\Sigma(\tau)_T \\
\mathsf{covered}_\Sigma(D)_T & := & \mathsf{covered}_\Sigma(D)_D \\
\mathsf{covered}_\Sigma(E)_T & := & \mathsf{covered}_\Sigma(E)_E \wedge \overrightarrow{E} \in \Sigma \\
\\
\mathsf{covered}_\Sigma(\mathbf{c})_D & := & \mathit{true} \\
\mathsf{covered}_\Sigma(D\,\mathcal{M})_D & := & \mathsf{covered}_\Sigma(D)_D \wedge \mathsf{covered}_\Sigma(\mathcal{M})_T
\end{array}
\qquad
\begin{array}{rcl}
\mathsf{covered}_\Sigma(\mathbb{e}\,\tau\,\bar{B})_G & := & \mathsf{covered}_\Sigma(\tau)_T \wedge \mathsf{covered}_\Sigma(\bar{B})_B \\
\mathsf{covered}_\Sigma(G\,\tau)_G & := & \mathsf{covered}_\Sigma(G)_G \wedge \mathsf{covered}_\Sigma(\tau)_T \\
\\
\mathsf{covered}_\Sigma(\langle G\rangle)_E & := & \mathsf{covered}_\Sigma(G)_G \\
\mathsf{covered}_\Sigma(E\,\tau)_E & := & \mathsf{covered}_\Sigma(E)_E \wedge \mathsf{covered}_\Sigma(\tau)_T \\
\\
\mathsf{covered}_\Sigma(\bar{\mathcal{M}} \Rightarrow \tau)_B & := & \mathsf{covered}_\Sigma(\bar{\mathcal{M}})_T \wedge \mathsf{covered}_\Sigma(\tau)_T \\
\\
\mathsf{covered}_\Sigma(\mathcal{M})_M & := & \bigcup_{\tau\in\mathcal{M}} \mathsf{covered}_\Sigma(\tau)_T
\end{array}
$$

- $\mathbf{c} \notin \mathcal{N}(\Sigma)$ *and* $\mathbf{c} \notin \mathcal{N}(\tau)$*; and*
- $\mathsf{covered}_\Sigma(\Gamma)$ *and* $\mathsf{covered}_\Sigma(\tau)$*.*

The proof of Step 1, namely that terms definable in λ_{e} terms are typable (Thm. 25), consists of two steps. First we show that $\to_{\mathsf{e}}$-normal forms are typable with good typing judgements. These typing judgements $\Gamma; \Sigma \vdash t : \tau$, for $t \in \mathsf{NF}(\to_{\mathsf{e}})$, are such that constants do not occur negatively in τ nor in Σ nor positively in Γ. However, constants may occur positively in τ such as when typing the normal form $\mathbf{c}$ and also negatively in Γ such as when typing the normal form $x\,\mathbf{c}$ where $x : [[\mathbf{c}] \to \alpha]$.

Lemma 23 (Normal forms are typable)**.** *Let* $t \in \mathsf{NF}(\to_{\mathsf{e}})$*. Then there exists a context* Γ*, an error context* Σ *and a type* τ *such that* $\pi(\Gamma; \Sigma \vdash t : \tau)$*, and* $\Gamma; \Sigma \vdash t : \tau$ *is good. Moreover, if* t *is of the form:*

- $x\,\bar{N}$*, then* τ *is a type variable; and*
- $(\mathtt{case}\ N_0\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow N_i)_{i\in I})\bar{N}$*, where* N_0 *does not match with* $(\mathbf{c}_i\bar{x}_i \Rightarrow N_i)_{i\in I}$*, then* $\tau = \langle \mathbb{e}\,\tau\,(\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i\in I}\,\rho_1 \ldots \rho_k\rangle$ *with* $k = |\bar{N}|$*.*

Proof. By induction on the structure of t, according to the characterization of Lem. 4. We thus consider three cases for t:

- $t = \lambda\bar{x}.y\bar{N}$ where $\bar{N} = N_1, \ldots, N_k$. By the *i.h.* there exist contexts $(\Gamma_i)_{i\in 1..k}$, error contexts $(\Sigma_i)_{i\in 1..k}$, types $(\tau_i)_{i\in 1..k}$ s.t. the typing judgements $\Gamma_i; \Sigma_i \vdash \tau_i$ are good, and derivations π_i s.t. $\pi_i(\Gamma_i; \Sigma_i \vdash N_i : \tau_i)_{i\in 1..k}$. Let $\Upsilon := \bigcup \Sigma_i$. Note that $\Gamma_i; \Sigma_i \vdash \tau_i$ good implies $\Gamma_i; \Upsilon \vdash \tau_i$ good. Also, $\Gamma_i; \Sigma_i \vdash N_i : \tau_i$ derivable implies $\Gamma_i; \Upsilon \vdash N_i : \tau_i$ derivable. Let α be a fresh type variable; we can build the following derivation:

$$\sum \Gamma_i + y : [[\tau_1] \to [\tau_2] \to \ldots \to [\tau_k] \to \alpha]; \Upsilon \vdash y\bar{N} : \alpha$$

Also, note that if $\bar{x}$ is empty, then this proves item (1). Let $\Delta := \sum \Gamma_i + y : [[\tau_1] \to [\tau_2] \to \ldots \to [\tau_k] \to \alpha]$. We can use TAbs to deduce

$$\Delta \setminus \bar{x}; \Upsilon \vdash \lambda\bar{x}.y\bar{N} : \Delta(x_1) \to \ldots \to \Delta(x_k) \to \alpha$$

$[] \notin \mathcal{P}(\Delta \setminus \bar{x}; \Upsilon \vdash \Delta(x_1) \to \ldots \to \Delta(x_k) \to \alpha)$ follows from $[] \notin \mathcal{P}(\Gamma_i; \Sigma_i \vdash \tau_i)$ above. Likewise, $\mathbf{c} \notin \mathcal{N}(\Delta \setminus \bar{x}; \Upsilon \vdash \Delta(x_1) \to \ldots \to \Delta(x_k) \to \alpha)$ follows from $\mathbf{c} \notin \mathcal{N}(\Gamma_i; \Sigma_i \vdash \tau_i)$ above.

- $t = \lambda\bar{x}.\mathbf{c}\bar{N}$ where $\bar{N} = N_1, \ldots, N_k$. By the *i.h.* there exist contexts $(\Gamma_i)_{i\in 1..k}$, error contexts $(\Sigma_i)_{i\in 1..k}$, types $(\tau_i)_{i\in 1..k}$ s.t. the typing judgements $\Gamma_i; \Sigma_i \vdash \tau_i$ are good, and derivations π_i s.t. $\pi_i(\Gamma_i; \Sigma_i \vdash N_i : \tau_i)_{i\in 1..k}$. Let $\Upsilon := \bigcup \Sigma_i$. Note that $\Gamma_i; \Sigma_i \vdash \tau_i$ good implies $\Gamma_i; \Upsilon \vdash \tau_i$ good. Also, $\Gamma_i; \Sigma_i \vdash N_i : \tau_i$ derivable implies $\Gamma_i; \Upsilon \vdash N_i : \tau_i$ derivable. Then we can build the derivation

  $$\sum \Gamma_i; \Upsilon \vdash \mathbf{c}\bar{N} : \mathbf{c}\,[\tau_1] \ldots [\tau_k]$$

  Let $\Delta := \sum \Gamma_i$. We can use TAbs to deduce

  $$\Delta \setminus \bar{x}; \Upsilon \vdash \lambda\bar{x}.\mathbf{c}\bar{N} : \Delta(x_1) \to \ldots \to \Delta(x_k) \to \mathbf{c}\,[\tau_1] \ldots [\tau_k]$$

  $[] \notin \mathcal{P}(\Delta \setminus \bar{x}; \Upsilon \vdash \Delta(x_1) \to \ldots \to \Delta(x_k) \to \mathbf{c}\,[\tau_1] \ldots [\tau_k])$ follows from $[] \notin \mathcal{P}(\Gamma_i; \Sigma_i \vdash \tau_i)$ above. Likewise, $\mathbf{c} \notin \mathcal{N}(\Delta \setminus \bar{x}; \Upsilon \vdash \Delta(x_1) \to \ldots \to \Delta(x_k) \to \mathbf{c}\,[\tau_1] \ldots [\tau_k])$ follows from $\mathbf{c} \notin \mathcal{N}(\Gamma_i; \Sigma_i \vdash \tau_i)$ above.

- $t = \lambda\bar{x}.(\texttt{case } M_0 \texttt{ of } (\mathbf{c}_i\bar{x}_i \Rightarrow M_i)_{i\in I})\bar{N}$, M_0 does not match with $(\mathbf{c}_i\bar{x}_i \Rightarrow M_i)_{i\in I}$, $\bar{N} = N_1, \ldots, N_k$ and we assume $I = 1..m$. By the *i.h.* there exist contexts $(\Gamma_i)_{i\in 1..k}$, error contexts $(\Sigma_i)_{i\in 1..k}$, types $(\tau_i)_{i\in 1..k}$ and derivations π_i s.t. $\pi_i(\Gamma_i; \Sigma_i \vdash N_i : \tau_i)_{i\in 1..k}$ and the typing judgements

  $$\Gamma_i; \Sigma_i \vdash N_i : \tau_i \text{ are good} \tag{5}$$

  The *i.h.* also allows us to deduce the existence of contexts $(\Delta_i)_{i\in 0..m}$, error contexts $(\Upsilon_i)_{i\in 0..m}$, types $(\sigma_i)_{i\in 0..m}$ and typing derivations ξ_i

  $$\xi_i(\Delta_i; \Upsilon_i \vdash M_i : \sigma_i)_{i\in 0..m} \tag{6}$$

  s.t. the typing judgements

  $$\Delta_i; \Upsilon_i \vdash M_i : \sigma_i \text{ are good} \tag{7}$$

  We consider multiple cases following Rem. 5:

  - M_0 is of the form $M_0 = x\,\bar{P}$. Then by item (1), σ_0 is a type variable α and hence $(\mathbf{c}_i\bar{x}_i \Rightarrow M_i)_{i\in I}$ follows from tCMismatch using (6):

$$\alpha \; \langle (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i\in I}\rangle \; (\sum_{i\in I} \Delta_i \setminus \bar{x}_i) \cup \{\rho\}; \Upsilon_i, \rho$$

where $\rho := \langle \mathbin{\text{e}} \alpha \, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i\in I}\rangle \tau_1 \dots \tau_k$. We then use tCase to deduce:

$$\Delta_0 + (\sum_{i\in I} \Delta_i \setminus \bar{x}_i); \bigcup \Upsilon_i \cup \{\rho\} \vdash \texttt{case}\ M_0\ \texttt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow M_i)_{i\in I} : \\ \langle \mathbin{\text{e}} \alpha \, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i\in I}\rangle \tau_1 \dots \tau_k$$

Multiple applications of tApp result in:

$$\Delta_0 + (\sum_{i\in I} \Delta_i \setminus \bar{x}_i) + \sum_{j\in 1..k} \Gamma_j; \bigcup \Upsilon_i \bigcup \Sigma_i \cup \{\rho\} \vdash (\texttt{case}\ M_0\ \texttt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow M_i)_{i\in I})\, \bar{N} : \\ \langle \mathbin{\text{e}} \alpha \, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i\in I}\, \tau_1 \dots \tau_k\rangle$$

Let $\Theta := \Delta_0 + (\sum_{i\in I} \Delta_i \setminus \bar{x}_i) + \sum_{j\in 1..k} \Gamma_j$ and $\Phi := \bigcup \Upsilon_i \cup \bigcup \Sigma_i \cup \{\rho\}$. Finally, we apply tAbs multiple times to deduce:

$$\Theta \setminus \bar{x}; \Phi \vdash \lambda \bar{x}.(\texttt{case}\ M_0\ \texttt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow M_i)_{i\in I})\, \bar{N} : \Theta(x_1) \to \dots \to \Theta(x_l) \to \\ \langle \mathbin{\text{e}} \alpha \, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i\in I}\, \tau_1 \dots \tau_k\rangle$$

Note that $\Theta \setminus \bar{x}; \Phi \vdash \Theta(x_1) \to \dots \to \Theta(x_l) \to \langle \mathbin{\text{e}} \alpha \, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i\in I}\, \tau_1 \dots \tau_k\rangle$ good follows from (5) and (7). In other words,

* $[] \notin \mathcal{P}(\Theta \setminus \bar{x}; \Phi \vdash \Theta(x_1) \to \dots \to \Theta(x_l) \to \langle \mathbin{\text{e}} \alpha \, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i\in I}\, \tau_1 \dots \tau_k\rangle)$; and
* $\mathbf{c} \notin \mathcal{N}(\Theta \setminus \bar{x}; \Phi \vdash \Theta(x_1) \to \dots \to \Theta(x_l) \to \langle \mathbin{\text{e}} \alpha \, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i\in I}\, \tau_1 \dots \tau_k\rangle)$

follows from (5) and (7). In particular, $\mathbf{c} \notin \mathcal{N}(\langle \mathbin{\text{e}} \alpha \, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i\in I}\rangle \tau_1 \dots \tau_k)$ and $[] \notin \mathcal{P}(\langle \mathbin{\text{e}} \alpha \, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i\in I}\rangle \tau_1 \dots \tau_k)$.

– M_0 is of the form $\lambda x.N$. Then following Lem. 18, σ_0 must be a functional type and hence we can proceed as in the previous case.

– M_0 is of the form $\mathbf{c}\bar{P}$. Since M_0 does not match with $(\mathbf{c}_i \bar{x}_i \Rightarrow M_i)_{i\in I}$, either $\mathbf{c} \notin (\mathbf{c}_i)_{i\in I}$ or $\mathbf{c} = \mathbf{c}_j$ for some $j \in I$ and $|\bar{P}| \neq |\bar{x}_j|$. Since $M_0 = \mathbf{c}\bar{P}$, then following Lem. 18 $\sigma_0 = \mathbf{c}\, \bar{\mathcal{M}}$ with $|\bar{\mathcal{M}}| = |\bar{P}|$. Thus σ_0 does not match $(\mathbf{c}_i \bar{x}_i \Rightarrow M_i)_{i\in I}$ and we can proceed as in the previous case.

– M_0 is of the form $(\texttt{case}\ N_0\ \texttt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow N_i)_{i\in I})\bar{P}$, where N_0 does not match with $(\mathbf{c}_i \bar{x}_i \Rightarrow N_i)_{i\in I}$. Then by the *i.h.* and item (2), $\sigma_0 = \mathbin{\text{e}} \rho \, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i\in I}$. Thus σ_0 does not match $(\mathbf{c}_i \bar{x}_i \Rightarrow M_i)_{i\in I}$ and we can proceed as in the previous case.

□

The second step consists of showing subject expansion for $\to_{\texttt{e}}$ (*i.e.* $t \to_{\texttt{e}} s$ and $\Gamma; \Sigma \vdash s : \tau$ imply $\Gamma; \Sigma \vdash t : \tau$).

Lemma 24 ($\to_{\texttt{e}}$-expansion)**.** *Let $t \to_{\texttt{e}} s$. If $\Gamma; \Sigma \vdash s : \tau$ then $\Gamma; \Sigma \vdash t : \tau$.*

Proof. By induction on the derivation of $\Gamma; \Sigma \vdash s : \tau$. We first present the cases in which reduction is at the root since these do not involve the *i.h.*.

- β-step. $t = (\lambda x.t_3)\, t_2$ and $t = (\lambda x.t_3) t_2 \to_{\texttt{e}} t_3\{x := t_2\} = s$. From Lem. 21 there exist Γ_1, Γ_2, and $\mathcal{M}$ such that $\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash t_3 : \tau$, $\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}$, $\Gamma = \Gamma_1 + \Gamma_2$. We can conclude by using tAbs and then tApp.

- $\texttt{fix}$-step. $t = \texttt{fix}(x.u) \to_{\texttt{e}} u\{x := \texttt{fix}(x.u)\} = s$. From Lem. 21 there exist Γ_1, Γ_2, and $\mathcal{M}$ such that $\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash u : \tau$, $\Gamma_2; \Sigma \vdash \texttt{fix}(x.u) : \mathcal{M}$, and $\Gamma = \Gamma_1 + \Gamma_2$. We can thus conclude immediately by using tFix.

- `case`-step. $t = \texttt{case}\ \mathbf{c}_j\bar{u}\ \texttt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \to_{\mathtt{e}} s_j\{\bar{x}_j := \bar{u}\} = s$ and $j \in I$ and $|\bar{u}| = |\bar{x}_j|$. Let $y = x_j$ so that $s_j\{\bar{x}_j := \bar{u}\}$ may be written $s_j\{\bar{y} := \bar{u}\}$. Also, assume $|\bar{y}| = n$. From Lem. 21 there exists Γ_{n1}, Γ_{n2}, Σ_{n1}, Σ_{n2} and $\mathcal{M}_n$ such that $\Gamma_{n1} \oplus y_n :: \mathcal{M}_n; \Sigma_{n1} \vdash s_j\{y_1 := u_1\}\ldots\{y_k := u_{n-1}\} : \tau$, $\Gamma_{n2}; \Sigma_{n2} \vdash t_2 : \mathcal{M}$, $\Gamma = \Gamma_{n1} + \Gamma_{n2}$ and $\Sigma = \Sigma_{n1} + \Sigma_{n2}$. Iterating further applications of Lem. 21 produces Γ_{11} and $\Gamma_{12}, \ldots, \Gamma_{(n-1)2}$, Σ_{11} and $\Sigma_{12}, \ldots, \Sigma_{(n-1)2}$ and $\mathcal{M}_1, \ldots, \mathcal{M}_{n-1}$ such that

  1. $\Gamma_{11} \oplus y_1 :: \mathcal{M}_1 \oplus \ldots \oplus y_n :: \mathcal{M}_n; \Sigma_{11} \vdash s_j : \tau$,
  2. $\Gamma_{i2}; \Sigma_{i2} \vdash u_i : \mathcal{M}_i$ (for $i \in 1..n$);
  3. $\Gamma = \Gamma_{11} + \sum_{i=1}^{n} \Gamma_{i2}$; and
  4. $\Sigma = \Sigma_{11} \cup \bigcup_{i=1}^{n} \Sigma_{i2}$.

  From the second item we deduce $\sum_{i=1}^{n} \Gamma_{i2}; \bigcup_{i=1}^{n} \Sigma_{i2} \vdash \mathbf{c}_j\,\bar{u} : \mathbf{c}_j\,\mathcal{M}_1 \ldots \mathcal{M}_n$. We are left to verify that $\mathbf{c}_j\,\mathcal{M}_1 \ldots \mathcal{M}_n\ \langle \mathbf{c}_1\bar{x}_1 \Rightarrow s_1 \ldots \mathbf{c}_n\bar{x}_n \Rightarrow s_n\rangle\ \Gamma_{11}; \Sigma_{11}, \tau$ in order to conclude that $\Gamma; \Sigma \vdash t : \tau$ using tCase. Since $\mathbf{c}_j\,\mathcal{M}_1 \ldots \mathcal{M}_n$ matches $\mathbf{c}_1\bar{x}_1 \Rightarrow s_1 \ldots \mathbf{c}_n\bar{x}_n \Rightarrow s_n$, we simply note that tCMatch applies since $\Gamma_{11} \oplus y_1 :: \mathcal{M}_1 \oplus \ldots \oplus y_n :: \mathcal{M}_n; \Sigma_{11} \vdash s_j : \tau$ is derivable.

Having concluded the cases in which reduction is at the root, we resume our proof of the rest of the cases by induction on the derivation of $\Gamma; \Sigma \vdash s : \tau$.

- tAbs. Then $t = \lambda x.u$, $u \to_{\mathtt{e}} u'$ and $s = \lambda x.u'$ and the derivation of $\Gamma; \Sigma \vdash \lambda x.u' : \tau$ ends as follows where $\tau = \mathcal{M} \to \sigma$:

$$\frac{\Gamma \oplus x :: \mathcal{M}; \Sigma \vdash u' : \sigma}{\Gamma; \Sigma \vdash \lambda x.u' : \mathcal{M} \to \sigma}\ \textsc{tAbs}$$

  From the *i.h.* we have $\Gamma \oplus x :: \mathcal{M}; \Sigma \vdash u : \sigma$, thus we may conclude using tAbs.

- tApp. Suppose $s = t_1'\,t_2$ with $t_1 \to_{\mathtt{e}} t_1'$ (the case in which $s = t_1\,t_2'$ with $t_2 \to_{\mathtt{e}} t_2'$ is similar an omitted). The derivation of $\Gamma; \Sigma \vdash t_1'\,t_2 : \tau$ must end as follows:

$$\frac{\Gamma_1; \Sigma \vdash t_1' : \sigma \qquad \sigma \mathbin{@} \mathcal{M} \Rightarrow \tau \qquad \Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}{\Gamma_1 + \Gamma_2; \Sigma \vdash t_1'\,t_2 : \tau}\ \textsc{tApp}$$

  where $\Gamma = \Gamma_1 + \Gamma_2$. We simply use the *i.h.* and then tApp.

- tFix. Then $s = \texttt{fix}(x.u')$ and $u \to_{\mathtt{e}} u'$. Moreover, the derivation of $\Gamma; \Sigma \vdash \texttt{fix}(x.u') : \tau$ must end as follows:

$$\frac{\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash u' : \tau \qquad \Gamma_2; \Sigma \vdash \texttt{fix}(x.u') : \mathcal{M}}{\Gamma_1 + \Gamma_2; \Sigma \vdash \texttt{fix}(x.u') : \tau}\ \textsc{tFix}$$

  We use the *i.h.* on the derivation of $\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash u' : \tau$ and conclude that $\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash u : \tau$ is derivable. Note that since $s = \texttt{fix}(x.u) \to_{\mathtt{e}} \texttt{fix}(x.u')$ and each derivation of $\Gamma_2; \Sigma \vdash \texttt{fix}(x.u') : \rho$, for $\rho \in \mathcal{M}$, is smaller than the given one, we can apply the *i.h.* to the derivation of $\Gamma_2; \Sigma \vdash \texttt{fix}(x.u') : \mathcal{M}$ and deduce $\Gamma_2; \Sigma \vdash \texttt{fix}(x.u) : \mathcal{M}$. We then conclude using tFix.

- tCase. $t = \texttt{case}\ u\ \texttt{of}\ \bar{b}$ where $\bar{b} = (\mathbf{c}_1\bar{x}_1 \Rightarrow s_1)\dots(\mathbf{c}_n\bar{x}_n \Rightarrow s_n)$. The derivation ends in:

$$\frac{\Gamma_1; \Sigma \vdash u : \sigma \qquad \sigma\ \langle\bar{b}\rangle\ \Gamma_2; \Sigma, \tau}{\Gamma_1 + \Gamma_2; \Sigma \vdash \texttt{case}\ u\ \texttt{of}\ \bar{b} : \tau}\ \text{tCase}$$

  where $\Gamma = \Gamma_1 + \Gamma_2$. If $u \to_{\mathtt{e}} u'$, then we simply apply the *i.h.* to the derivation of $\Gamma_1; \Sigma \vdash u : \sigma$ and then conclude using tCase. If $s_j \to_{\mathtt{e}} s'_j$, then if s_j occurs in a subderivation of the derivation of $\sigma\ \langle\bar{b}\rangle\ \Gamma_2; \Sigma, \tau$ we apply the *i.h.* and build a derivation of $\sigma\ \langle\bar{b'}\rangle\ \Gamma_2; \Sigma, \tau$ and conclude using tCase. Otherwise, we conclude immediately.

□

Theorem 25 (Step 1)**.** *Suppose t is definable in $\lambda_{\mathtt{e}}$. Then there exists a context Γ, an error context Σ, a type τ and a derivation π s.t. $\pi(\Gamma; \Sigma \vdash t : \tau)$ with $\Gamma; \Sigma \vdash t : \tau$ good.*

Proof. Suppose t is definable as s in $\lambda_{\mathtt{e}}$. Then $t \in \mathsf{WN}(\to_{\mathtt{e}})$. The result is an immediate consequence of Lem. 23 and Lem. 24. □

4.2 Typable Terms are Definable in $\lambda_{\mathtt{sh}}$ (Step 2)

The idea behind Step 2 is to show that: 1) redexes in a term t that are accounted for by a typing derivation for t, lets call them *typed-redexes*, are finite in number and that that number can only decrease by reducing them; and 2) terms that are in such *typed redex*-normal form and that are typed with good typing judgements are also in normal form with respect to $\lambda_{\mathtt{sh}}$ (*i.e.* are in $\mathsf{NF}(\to_{\mathtt{sh}})$). That a redex is accounted for in a typing derivation π is expressed as the redex occurring at a *typed occurrence* in π. For example, in a term such as $x\,(\mathtt{id}\,\mathtt{id})$ with $x : [\,] \to \alpha]$, the redex $\mathtt{id}\,\mathtt{id}$ is not typed-redex since there is no subderivation of π that types/accounts for it.

Positions in terms are defined as usual; we write ϵ for the empty position. Let p be a position in a term t and let $\pi(\Gamma; \Sigma \vdash t : \tau)$.

Definition 26. *We say p* **is a typed occurrence in** *π and define it inductively on π:*

- *π ends in* tVar *or* tCons*: $p = \epsilon$*
- *π ends in* tAbs*: Then π has the following form and $p = \epsilon$ or $p = 0.p'$ and p' is a typed occurrence in π_0.*

$$\frac{\dfrac{\pi_0}{\Gamma \oplus x :: \mathcal{M}; \Sigma \vdash t : \tau}}{\Gamma; \Sigma \vdash \lambda x.t : \mathcal{M} \to \tau}\ \text{tAbs}$$

- *π ends in* tApp*: Then π has the following form and $p = \epsilon$ or $p = 0.p'$ and p' is a typed occurrence in π_0 or $p = 1.p'$ and $\mathcal{M} \neq [\,]$ and p' is a typed occurrence in π_1.*

$$\frac{\Gamma; \Sigma \vdash t : \tau \quad \tau \mathbin{@} \mathcal{M} \Rightarrow \sigma \quad \Delta; \Sigma \vdash s : \mathcal{M}}{\Gamma\Delta; \Sigma \vdash ts : \sigma}\ \text{tApp}$$

- *π ends in* tFix*: Then π has the following form and $p = \epsilon$ or $p = 0.p'$ and p' is a typed occurrence in π_0.*

$$\dfrac{\dfrac{\pi_0}{\Gamma \oplus x :: \mathcal{M}; \Sigma \vdash t : \tau} \qquad \dfrac{\pi_1}{\Delta; \Sigma \vdash \mathtt{fix}(x.t) : \mathcal{M}}}{\Gamma\Delta; \Sigma \vdash \mathtt{fix}(x.t) : \tau}\ \textsc{tFix}$$

- *π ends in* tCase *with a matching case. Then π has the following form and $p = \epsilon$ or $p = 0.p'$ and p' is a typed occurrence in π_0 or $p = j.p'$ and p' is a typed occurrence in π_1.:*

$$\dfrac{\dfrac{\pi_0}{\Gamma; \Sigma \vdash t : \mathbf{c}_j\bar{\mathcal{M}}} \qquad \dfrac{\dfrac{\pi_1}{\Delta \oplus \bar{x}_j :: \bar{\mathcal{M}}; \Sigma \vdash s_j : \sigma_j}}{\mathbf{c}_j\bar{\mathcal{M}}\ \langle(\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}\rangle\ \Delta; \Sigma, \sigma_j}\ \textsc{tCMatch}}{\Gamma\Delta; \Sigma \vdash \mathtt{case}\ t\ \mathtt{of}\ \bar{b} : \sigma_j}\ \textsc{tCase}$$

and $\mathbf{c}_j\bar{\mathcal{M}}$ matches $(\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$.

- *π ends in* tCase *with a mismatching case: Then π has the following form and $p = \epsilon$ or $p = 0.p'$ and p' is a typed occurrence in π_0 or $p = j.p'$, with $j \in I$, and p' is a typed occurrence in π_j.:*

$$\dfrac{\dfrac{\pi_0}{\Gamma; \Sigma \vdash t : \tau} \qquad \dfrac{\dfrac{\pi_j}{\left(\Gamma_i \oplus \bar{x}_i :: \bar{\mathcal{M}}_i; \Sigma \vdash s_i : \sigma_i\right)_{i\in I}} \qquad \rho := \langle \mathtt{e}\,\tau\,(\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i\in I}\rangle\,\rho_1 \ldots \rho_k}{\tau\ \langle(\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}\rangle\ \Delta; \Sigma \cup \{\rho\}, \rho}\ \textsc{tCMismatch}}{\Gamma\Delta; \Sigma \cup \{\rho\} \vdash \mathtt{case}\ t\ \mathtt{of}\ \bar{b} : \sigma}\ \textsc{tCase}$$

and τ does not match $(\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ and $\Delta = (\sum_{i\in I}\Gamma_i)$.

- *π ends in* tES*: Then π has the following form and $p = \epsilon$ or $p = 0.p'$ and p' is a typed occurrence in π_0 or $p = 1.p'$ and p' is a typed occurrence in π_1.*

$$\dfrac{\dfrac{\pi_0}{\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash t : \tau} \qquad \dfrac{\pi_1}{\Gamma_2; \Sigma \vdash s : \mathcal{M}}}{\Gamma_1\Gamma_2; \Sigma \vdash t[x\backslash s] : \tau}\ \textsc{tES}$$

- *π ends in* tMulti*: Then π has the following form and $p = \epsilon$ or $p = i.p'$ and p' is a typed occurrence in π_i.*

$$\dfrac{\dfrac{\pi_i}{(\Gamma_i; \Sigma \vdash t : \tau_i)_{1\le i\le n}} \qquad (n \ge 0)}{\sum_{i=1}^{n}\Gamma_i; \Sigma \vdash t : \sum_{i=1}^{n}[\tau_i]}\ \textsc{tMulti}$$

We say t is in **π-normal form** iff t has no typed redex occurrences in π. We write $\mathtt{size}(\pi)$ to denote the size of a derivation in terms of the number of nodes of the derivation seen as a tree.

Similarly, $\mathtt{size}(\Pi)$ is the sum of the sizes of all the derivations in the set of derivations Π. Also, $\mathtt{fix}(\pi)$ denotes the number of nodes in π that are instances of $\textsc{tFix}$.

The **measure** of a derivation π, $\mathtt{M}(\pi)$, is the pair $\langle \mathtt{size}(\pi), \mathtt{fix}(\pi)\rangle$. The following results all aim at showing that this measure decreases when typed redex occurrences are reduced (*cf.* Lem. 31). We begin our analysis by adding two typing rules to $\mathcal{T}$ that allow us to type substitution contexts. This will be a convenient technical device to analyse the typing of terms of the form $t\mathtt{L}$ (*cf.* Lem. 27 below).

$$\frac{}{\varnothing; \Sigma \Vdash \epsilon : \varnothing}\ \textsc{tCHole}$$

$$\frac{\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \Vdash \mathtt{L} : \Delta \quad x \notin \operatorname{dom}\Delta \quad \Gamma_2; \Sigma \vdash s : \mathcal{M} \oplus \mathcal{N}}{\Gamma_1\Gamma_2; \Sigma \Vdash \mathtt{L}[x\backslash s] : \Delta \oplus x : \mathcal{N}}\ \textsc{tCCons}$$

Lemma 27. *$\pi_{t\mathtt{L}}(\Gamma; \Sigma \vdash t\mathtt{L} : \tau)$ iff $\exists \Gamma_1, \Gamma_2, \Gamma_3$ s.t. $\Gamma = \Gamma_1\Gamma_2$, and $\pi_{\mathtt{L}}(\Gamma_1; \Sigma \Vdash \mathtt{L} : \Gamma_3)$ and $\pi_t(\Gamma_2\Gamma_3; \Sigma \vdash t : \tau)$ and* $\mathtt{size}(\pi_{t\mathtt{L}}) = \mathtt{size}(\pi_{\mathtt{L}}) + \mathtt{size}(\pi_t) - 1$.

Proof. By induction on $\mathtt{L}$. If $\mathtt{L} = \epsilon$, then we set $\Gamma_1 := \varnothing$, $\Gamma_3 := \varnothing$, and $\Gamma_2 := \Gamma$. Suppose $\mathtt{L} = \mathtt{L}'[x\backslash s]$. Then $\pi_{t\mathtt{L}}$ has the form:

$$\frac{\dfrac{\pi_{t\mathtt{L}'}}{\Delta_1 \oplus x :: \mathcal{M}; \Sigma \vdash t\mathtt{L}' : \tau} \qquad \dfrac{\Pi_s}{\Delta_2; \Sigma \vdash s : \mathcal{M}}}{\Delta_1\Delta_2; \Sigma \vdash t\mathtt{L}'[x\backslash s] : \tau}\ \textsc{tES}$$

where $\Gamma = \Delta_1\Delta_2$. By the *i.h.* on $\pi_{t\mathtt{L}'}$, $\exists \Gamma_{11}, \Gamma_{12}, \Gamma_{13}$ s.t. $\Delta_1 \oplus x :: \mathcal{M} = \Gamma_{11}\Gamma_{12}$, and $\pi_{\mathtt{L}'}(\Gamma_{11}; \Sigma \Vdash \mathtt{L}' : \Gamma_{13})$ and $\pi_t(\Gamma_{12}\Gamma_{13}; \Sigma \vdash t : \tau)$ and $\mathtt{size}(\pi_{t\mathtt{L}'}) = \mathtt{size}(\pi_{\mathtt{L}'}) + \mathtt{size}(\pi_t) - 1$. From $\Delta_1 \oplus x :: \mathcal{M} = \Gamma_{11}\Gamma_{12}$ we deduce:

$$\Delta_1 \oplus x :: \mathcal{M} = \Gamma_{11} + \Gamma_{12} = \Gamma'_{11} \oplus x :: \mathcal{M}_1 + \Gamma'_{12} \oplus x :: \mathcal{M}_2 \tag{8}$$

where $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$. We next construct the derivation $\pi_{\mathtt{L}'[x\backslash s]}$:

$$\frac{\Gamma'_{11} \oplus x :: \mathcal{M}_1; \Sigma \Vdash \mathtt{L}' : \Gamma_{13} \quad x \notin \operatorname{dom}\Gamma_{13} \quad \Delta_2; \Sigma \vdash s : \mathcal{M}}{\Gamma'_{11}\Delta_2; \Sigma \Vdash \mathtt{L}'[x\backslash s] : \Gamma_{13} \oplus x :: \mathcal{M}_2}\ \textsc{tCCons}$$

We set

$$\Gamma_1 := \Gamma'_{11}\Delta_2 \quad \Gamma_2 := \Gamma'_{12} \quad \Gamma_3 := \Gamma_{13} \oplus x :: \mathcal{M}_2 \tag{9}$$

Note that $\Gamma = \Delta_1\Delta_2 = \Gamma_1\Gamma_2$. Moreover,

$$\begin{array}{ll} & \mathtt{size}(\pi_{t\mathtt{L}}) \\ = & \mathtt{size}(\pi_{t\mathtt{L}'}) + \mathtt{size}(\Pi_s) + 1 \\ = & \mathtt{size}(\pi_{\mathtt{L}'}) + \mathtt{size}(\pi_t) - 1 + \mathtt{size}(\Pi_s) + 1 \\ = & (\mathtt{size}(\pi_{\mathtt{L}'}) + \mathtt{size}(\Pi_s) + 1) + \mathtt{size}(\pi_t) - 1 \\ = & \mathtt{size}(\pi_{\mathtt{L}'[x\backslash s]}) + \mathtt{size}(\pi_t) - 1 \end{array}$$

$\square$

Lemma 28. *If $\pi(\Gamma; \Sigma \vdash (\lambda x.t_1)\mathtt{L}\, t_2 : \tau)$, then there exists π' s.t. $\pi'(\Gamma; \Sigma \vdash t_1[x\backslash t_2]\mathtt{L} : \tau)$ and* $\mathtt{size}(\pi) > \mathtt{size}(\pi')$.

Proof. By induction on $\mathtt{L}$.

- Suppose $\mathtt{L} = \epsilon$. Then π has the form:

$$\cfrac{\cfrac{\pi_1}{\Gamma_1; \Sigma \vdash \lambda x.t_1 : \sigma} \qquad \sigma \mathbin{@} \mathcal{M} \Rightarrow \tau \qquad \cfrac{\Pi_2}{\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}\ \textsc{tMulti}}{\Gamma_1\Gamma_2; \Sigma \vdash (\lambda x.t_1)t_2 : \tau}\ \textsc{tApp}$$

By Lem. 20(2) on π_1, $\sigma = \mathcal{M} \to \tau$. Moreover, by Lem. 18(2), π_1 must end with $\textsc{tAbs}$:

$$\cfrac{\cfrac{\cfrac{\pi_{11}}{\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash t_1 : \tau}}{\Gamma_1; \Sigma \vdash \lambda x.t : \mathcal{M} \to \tau}\ \textsc{tAbs} \qquad \mathcal{M} \to \tau \mathbin{@} \mathcal{M} \Rightarrow \tau \qquad \cfrac{\Pi_2}{\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}\ \textsc{tMulti}}{\Gamma_1\Gamma_2; \Sigma\ \vdash (\lambda x.t_1)t_2 : \tau}\ \textsc{tApp}$$

We construct the following derivation π'

$$\cfrac{\cfrac{\pi_{11}}{\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash t_1 : \tau} \qquad \cfrac{\Pi_2}{\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}\ \textsc{tMulti}}{\Gamma_1\Gamma_2; \Sigma \vdash t_1[x \backslash t_2] : \tau}\ \textsc{tES}$$

Note that $\mathtt{size}(\pi) = \mathtt{size}(\pi_{11}) + 1 + \mathtt{size}(\Pi_2) + 2 > \mathtt{size}(\pi_{11}) + \mathtt{size}(\Pi_2) + 2 = \mathtt{size}(\pi')$.

- For the inductive case, suppose $\mathtt{L} = \mathtt{L}'[y \backslash s]$. Then π has the form:

$$\cfrac{\cfrac{\pi_1}{\Gamma_1; \Sigma\ \vdash (\lambda x.t_1)\mathtt{L}'[y \backslash s] : \sigma} \qquad \sigma \mathbin{@} \mathcal{M} \Rightarrow \tau \qquad \cfrac{\Pi_2}{\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}}{\Gamma_1\Gamma_2; \Sigma\ \vdash (\lambda x.t_1)\mathtt{L}'[y \backslash s]t_2 : \tau}\ \textsc{tApp}$$

By Lem. 18(2), there exist Γ_{11}, Γ_{12} s.t. $\Gamma_1 = \Gamma_{11}\Gamma_{12}$, and π_1 must end with $\textsc{tES}$:

$$\cfrac{\cfrac{\cfrac{\pi_{11}}{\Gamma_{11} \oplus y :: \mathcal{N}; \Sigma \vdash (\lambda x.t_1)\mathtt{L}' : \sigma} \qquad \cfrac{\Pi_3}{\Gamma_{12}; \Sigma \vdash s : \mathcal{N}}}{\Gamma_1; \Sigma \vdash (\lambda x.t_1)\mathtt{L}'[y \backslash s] : \sigma} \qquad \sigma \mathbin{@} \mathcal{M} \Rightarrow \tau \qquad \cfrac{\Pi_2}{\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}}{\Gamma_1\Gamma_2; \Sigma \vdash (\lambda x.t_1)\mathtt{L}'[y \backslash s]t_2 : \tau}\ \textsc{t}$$

We can then build the following derivation π_2:

$$\cfrac{\cfrac{\pi_{11}}{\Gamma_{11} \oplus y :: \mathcal{N}; \Sigma \vdash (\lambda x.t_1)\mathtt{L}' : \sigma} \qquad \sigma \mathbin{@} \mathcal{M} \Rightarrow \tau \qquad \cfrac{\Pi_2}{\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}}{\Gamma_1\Gamma_2; \Sigma \vdash (\lambda x.t_1)\mathtt{L}'t_2 : \sigma}\ \textsc{tApp}$$

and apply the *i.h.* to it to deduce that there exists π_2' s.t. $\pi_2'(\Gamma; \Sigma \vdash t_1[x \backslash t_2]\mathtt{L}' : \sigma)$ and $\mathtt{size}(\pi_2) > \mathtt{size}(\pi_2')$. Finally, we construct the derivation π':

$$\dfrac{\dfrac{\pi_2'}{\Gamma \oplus y :: \mathcal{N}; \Sigma \vdash t_1[x\backslash t_2]\mathtt{L}' : \sigma} \quad \dfrac{\Pi_2}{\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}}{\Gamma_1\Gamma_2; \Sigma \vdash t_1\mathtt{L}'[y\backslash s]t_2 : \tau} \text{ TES}$$

Note that:

$$\begin{array}{rl} & \mathtt{size}(\pi) \\ = & \mathtt{size}(\pi_1) + \mathtt{size}(\Pi_2) + 2 \\ = & \mathtt{size}(\pi_{11}) + \mathtt{size}(\Pi_3) + 2 + \mathtt{size}(\Pi_2) + 2 \\ = & \mathtt{size}(\pi_{11}) + \mathtt{size}(\Pi_2) + 2 + \mathtt{size}(\Pi_3) + 2 \ . \\ = & \mathtt{size}(\pi_2) + \mathtt{size}(\Pi_3) + 2 \\ > & \mathtt{size}(\pi_2') + \mathtt{size}(\Pi_3) + 2 \\ = & \mathtt{size}(\pi') \end{array}$$

□

Lemma 29. *If* $\pi_1(\Gamma_1; \Sigma \vdash \mathtt{A}[\mathbf{c}_j] : \mathbf{c}\,\mathcal{M}_1 \ldots \mathcal{M}_n)$*, and* $\bar{\mathcal{M}} = \mathcal{M}_1, \ldots, \mathcal{M}_n$ *and* $\pi_2(\Gamma_2 \oplus \bar{x} :: \bar{\mathcal{M}}; \Sigma \vdash t : \sigma)$ *and* $|\mathtt{A}| = |\bar{x}|$*, then* $\pi_3(\Gamma_1 + \Gamma_2; \Sigma \vdash t[\bar{x}\backslash\mathtt{A}] : \sigma)$*. Moreover,* $\mathtt{size}(\pi_3) = \mathtt{size}(\pi_2) + \mathtt{size}(\pi_1) - 1$*.*

Proof. By induction on $\mathtt{A}$ using Lem. 27. □

Lemma 30. *If* $\pi_{\mathtt{C}[x]}(\Gamma \oplus x :: [\sigma_i]_{i \in I}; \Sigma \vdash \mathtt{C}[x] : \tau)$ *and* $\pi_s^i(\Delta_i; \Upsilon_i \vdash s : \sigma_i)_{i \in I}$*, then*

1. *for some* $K \subseteq I$*:*

$$\pi_{\mathtt{C}[s]}(\Gamma \oplus x :: [\sigma_i]_{i \in I \setminus K} + \sum_{k \in K} \Delta_k; \Sigma \cup \bigcup_{k \in K} \Upsilon_k \vdash \mathtt{C}[s] : \tau)$$

   where $\mathtt{size}(\pi_{\mathtt{C}[s]}) = \mathtt{size}(\pi_{\mathtt{C}[x]}) + \sum_{k \in K} \mathtt{size}(\pi_s^k) - |K|$*.*

2. *Moreover, if* $p \in \mathtt{pos}(\mathtt{C})$ *is the occurrence of the hole in* $\mathtt{C}$ *and* p *is a typed-occurrence in* $\pi_{\mathtt{C}[x]}$*, then* $K \neq \varnothing$*.*

Proof. By induction on $\mathtt{C}$. □

Lemma 31 (Weighted Subject Reduction for $\mathtt{sh}$)**.** *Let* $\pi(\Gamma; \Sigma \vdash t : \tau)$*. If* $t \to_{\mathtt{sh}} t'$*, then there exists* π' *such that* $\pi'(\Gamma; \Sigma \vdash t' : \tau)$*. Moreover, if this step reduces a typed* $\to_{\mathtt{sh}}$*-redex occurrence of* t *in* π*, then either*

1. $\mathtt{size}(\pi) > \mathtt{size}(\pi')$*; or*
2. $\mathtt{size}(\pi) = \mathtt{size}(\pi')$ *and* $\mathtt{fix}(\pi) > \mathtt{fix}(\pi')$

Proof. By induction on $t \to_{\mathtt{sh}} t'$. We first consider the case in which the reduction step takes place at the root of the term.

- $t = (\lambda x.t_1)\mathtt{L}\, t_2 \mapsto_{\mathtt{dB}} t_1[x\backslash t_2]\mathtt{L} = t'$. Immediate from Lem. 28.

- $t = \mathtt{C}[\![x]\!][x\backslash v\mathtt{L}] \mapsto_{\mathtt{lsv}} \mathtt{C}[v][x\backslash v]\mathtt{L} = t'$. The derivation π ends in:

$$\dfrac{\dfrac{\pi_{\mathtt{C}[\![x]\!]}}{\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash \mathtt{C}[\![x]\!] : \tau} \qquad \dfrac{\Pi_{v\mathtt{L}}}{\Gamma_2; \Sigma \vdash v\mathtt{L} : \mathcal{M}}}{\Gamma_1\Gamma_2; \Sigma \vdash \mathtt{C}[\![x]\!][x\backslash v\mathtt{L}] : \tau}\ \text{TES}$$

Let $\mathcal{M} = [\sigma_i]_{i\in I}$ and $\Pi_2 = \{\xi_i\}_{i\in I}$. Then $\Gamma_2 = \sum_{i\in I}\Gamma_{2i}$. Thus for each $\xi_i \in \Pi_{v\mathtt{L}}$, by Lem. 27, there exist $\Gamma_{2i1}, \Gamma_{2i2}, \Gamma_{2i3}$ s.t. $\Gamma_{2i} = \Gamma_{2i1}\Gamma_{2i2}$, and $\pi^i_{\mathtt{L}}(\Gamma_{2i1}; \Sigma \Vdash \mathtt{L} : \Gamma_{2i3})$ and $\pi^i_v(\Gamma_{2i2}\Gamma_{2i3}; \Sigma \vdash v : \sigma_i)$.

From the derivation $\pi_{\mathtt{C}[\![x]\!]}$ and Lem. 30, we deduce

$$\pi_{\mathtt{C}[v]}(\Gamma_1 \oplus x :: \mathcal{N} + \sum_{k\in K}\Gamma_{2k}; \Sigma \vdash \mathtt{C}[v] : \tau)$$

where $\mathcal{N} = [\sigma_i]_{i\in I\backslash K}$ for some $K \subseteq I$ and where

$$\mathtt{size}(\pi_{\mathtt{C}[v]}) = \mathtt{size}(\pi_{\mathtt{C}[x]}) + \sum_{k\in K}\mathtt{size}(\xi_k) - |K| \tag{10}$$

We thus construct the following derivation π':

$$\dfrac{\pi_{\mathtt{C}[v]} \qquad \dfrac{\Pi_3}{\sum_{k\in I\backslash K}\Gamma_{2k}; \Sigma \vdash v\mathtt{L} : \mathcal{N}}}{\Gamma_1\Gamma_2; \Sigma \vdash \mathtt{C}[\![v]\!][x\backslash v\mathtt{L}] : \tau}\ \text{TES}$$

where $\Pi_3 = \{\xi_i\}_{i\in I\backslash K}$. This concludes the first item of the lemma. We now address the second one.

$$\begin{array}{ll}
 & \mathtt{size}(\pi') \\
= & \mathtt{size}(\pi_{\mathtt{C}[v]}) + \mathtt{size}(\Pi_3) + 2 \\
= & \mathtt{size}(\pi_{\mathtt{C}[v]}) + \sum_{i\in I\backslash K}\mathtt{size}(\xi_i) + 2 \\
=_{(10)} & \mathtt{size}(\pi_{\mathtt{C}[x]}) + \sum_{j\in K}\mathtt{size}(\xi_j) - |K| + \sum_{i\in I\backslash K}\mathtt{size}(\xi_i) + 2 \\
= & \mathtt{size}(\pi_{\mathtt{C}[x]}) + \mathtt{size}(\Pi_2) + 2 - |K| \\
= & \mathtt{size}(\pi) - |K| \\
<_{Lem.\ 30(2)} & \mathtt{size}(\pi)
\end{array}$$

- $t = t_1[x\backslash t_2] \mapsto_{\mathtt{gc}} t_1 = t'$, with $x \notin \mathsf{fv}(t_1)$. The derivation of t has the following form:

$$\dfrac{\dfrac{\pi_1}{\Gamma \oplus x :: \mathcal{M}; \Sigma \vdash t_1 : \tau} \qquad \dfrac{\Pi_2}{\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}}{\Gamma_1\Gamma_2; \Sigma \vdash t_1[x\backslash t_2] : \tau}\ \text{TES}$$

By Lem. 19, $\mathcal{M} = []$. Then we set π' to be π_1 and conclude that the first item of this lemma holds: $\mathtt{size}(\pi') = \mathtt{size}(\pi_1) < \mathtt{size}(\pi_1) + 1 = \mathtt{size}(\pi)$.

- $t = \texttt{fix}(x.t_1) \mapsto_{\texttt{fix}} t_1[x\backslash\texttt{fix}(x.t_1)]$. Then π must have the form:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash t_1 : \tau} \qquad \dfrac{\Pi_1}{\Gamma_2; \Sigma \vdash \texttt{fix}(x.t_1) : \mathcal{M}}}{\Gamma_1\Gamma_2; \Sigma \vdash \texttt{fix}(x.t_1) : \tau} \text{ tFix}$$

Then we construct the following derivation π':

$$\dfrac{\dfrac{\pi_1}{\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash t_1 : \tau} \qquad \dfrac{\Pi_1}{\Gamma_2; \Sigma \vdash \texttt{fix}(x.t_1) : \mathcal{M}}}{\Gamma_1\Gamma_2; \Sigma \vdash t_1[x\backslash\texttt{fix}(x.t_1)] : \tau} \text{ tES}$$

The second item of the lemma holds, namely $\texttt{size}(\pi') = \texttt{size}(\pi)$ and $\texttt{fix}(\pi') < \texttt{fix}(\pi)$.

- $t = \texttt{case}\ \texttt{A}[\mathbf{c}_j]\texttt{L}\ \texttt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \mapsto_{\texttt{case}} s_j[\bar{x}_j\backslash\texttt{A}]\texttt{L} = t'$, where $|\texttt{A}[\mathbf{c}_j]\texttt{L}| = |\bar{x}_j|$ and $j \in I$. Then π must have the following form, where $\bar{b}$ abbreviates $(\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ and $\tau = \sigma_j$:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1; \Sigma_1 \vdash \texttt{A}[\mathbf{c}_j]\texttt{L} : \mathbf{c}_j\bar{\mathcal{M}}} \qquad \dfrac{\mathbf{c}_j\bar{\mathcal{M}} \text{ matches } \bar{b} \qquad \dfrac{\pi_2}{\Gamma_2 \oplus \bar{x}_j :: \bar{\mathcal{M}}; \Sigma_2 \vdash s_j : \sigma_j}}{\mathbf{c}_j\bar{\mathcal{M}}\ \langle\bar{b}\rangle\ \Gamma_2; \Sigma_2, \sigma_j} \text{ tCMatch}}{\Gamma_1\Gamma_2; \Sigma_1\Sigma_2 \vdash \texttt{case}\ \texttt{A}[\mathbf{c}_j]\texttt{L}\ \texttt{of}\ \bar{b} : \sigma_j} \text{ tCase}$$

By Lem. 29 on π_1 we can construct a derivation $\pi'(\Gamma_1\Gamma_2; \Sigma_1\Sigma_2 \vdash s_j[\bar{x}\backslash\texttt{A}]\texttt{L} : \sigma_j)$ s.t. $\texttt{size}(\pi') = \texttt{size}(\pi_2) + \texttt{size}(\pi_1) - 1 < \texttt{size}(\pi_2) + \texttt{size}(\pi_1) + 2 = \texttt{size}(\pi)$.

For the inductive cases (i.e. when $t = \lambda x.t_1$, $t = t_1\, t_2$, $t = \texttt{fix}(x.t_1)$, $t = \texttt{case}\ t_1\ \texttt{of}\ \bar{b}$, or $t = t_1[x\backslash t_2]$), we conclude from the *i.h.*.

□

The following result is not an immediate consequence of Lem. 20 since our type system does not have unique types.

Lemma 32 (Constant Answers and their Type)**.** *If* $\pi(\Gamma; \Sigma \vdash \texttt{A}[\mathbf{c}]\texttt{L} : \mathbf{d}\,\mathcal{M}_1 \ldots \mathcal{M}_n)$, *then* $\mathbf{c} = \mathbf{d}$ *and* $n = |\texttt{A}|$.

Proof. By induction on π. □

Lemma 33. *If* $\pi(\Gamma; \Sigma \vdash t : \tau)$ *and* t *is a weak structure (*i.e. $t = \texttt{E}[x]$*), then* $\Gamma(x) \neq []$.

Proof. By induction on π.

- TVar. The result is immediate.
- tAbs, tCons, tFix and tCase. The result is immediate since abstractions, constants, fixed point and case expressions are not weak structures.

- tApp. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1;\Sigma \vdash t_1 : \sigma} \qquad \sigma @ \mathcal{M} \Rightarrow \tau \qquad \Gamma_2;\Sigma \vdash t_2 : \mathcal{M}}{\Gamma_1\Gamma_2;\Sigma \vdash t_1 t_2 : \tau}\ \text{tApp}$$

Then $\mathtt{E} = \mathtt{E}_1\, t_2$ and $\mathtt{E}_1[x] = t_1$ and we conclude from the *i.h.* on π_1.

- tES. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1 \oplus y :: \mathcal{M};\Sigma\ \vdash t_1 : \tau} \qquad \dfrac{\Pi_2}{\Gamma_2;\Sigma \vdash t_2 : \mathcal{M}}}{\Gamma_1\Gamma_2;\Sigma \vdash t_1[y\backslash t_2] : \tau}\ \text{tES}$$

We consider two cases:

  – $\mathtt{E} = \mathtt{E}_1[y\backslash t_2]$ with $x \neq y$. We conclude from the *i.h.* on π_1.
  – $\mathtt{E} = \mathtt{E}_1[\![y]\!][y\backslash \mathtt{E}_2]$. From the *i.h.* on π_1 we deduce $\mathcal{M} \neq []$. Thus we can conclude from the *i.h.* on any $\pi_2 \in \Pi_2$.

□

Lemma 34 (Derivation Normal Forms)**.** *If $\pi(\Gamma;\Sigma \vdash t : \tau)$ and t is π-normal, then one of the following holds:*

- *t is an answer*
- *t is a weak structure (i.e. $t = \mathtt{E}[x]$)*
- *t is a weak error term (i.e. $t = \mathtt{F}[\mathtt{case}\ s\ \mathtt{of}\ \bar{b}]$, s is an answer or a weak structure, and $s \not\succ \bar{b}$).*

Proof. By induction on π.

- tVar. Then $\mathtt{E} = \Box$ and t is a weak structure.
- tAbs and tCons. Then t is an answer.
- tApp. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1;\Sigma \vdash t_1 : \sigma} \qquad \sigma @ \mathcal{M} \Rightarrow \tau \qquad \Gamma_2;\Sigma \vdash t_2 : \mathcal{M}}{\Gamma_1\Gamma_2;\Sigma \vdash t_1\, t_2 : \tau}\ \text{tApp}$$

Note that t_1 is π_1-normal. By the *i.h.*, t_1 is either a:

  – weak structure with weak context $\mathtt{E}'$ and we set $\mathtt{E} = \mathtt{E}'\, t_2$ and conclude; or
  – weak error term with weak context $\mathtt{F}'$ and we set $\mathtt{F} = \mathtt{F}'\, t_2$ and conclude; or
  – constant answer (it cannot be an abstraction answer since otherwise t would not be π-normal) and we conclude directly since then t is a constant answer.

- tFix. The result holds vacuously since t is assumed π-normal.
- tCase. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1;\Sigma \vdash s : \sigma} \qquad \sigma\ \langle \bar{b} \rangle\ \Gamma_2;\Sigma,\tau}{\Gamma_1\Gamma_2;\Sigma \vdash \texttt{case}\ s\ \texttt{of}\ \bar{b} : \tau}\ \text{tCase}$$

Note that s is π_1-normal. By the *i.h.* one of the following cases holds:

  - s is an answer. By Lem. 20, s can take one of two forms:
    - $s = \mathtt{A}[\mathbf{c}]\mathtt{L}$. Then $\sigma = \mathbf{c}\,\mathcal{M}_1 \ldots \mathcal{M}_n$ where $n = |\mathtt{A}[\mathbf{c}]\mathtt{L}|$. Since s does not enable $\bar{b}$, then it must be the case that σ does not match $\bar{b}$.
    - $s = (\lambda x.s')\mathtt{L}$. Then $\sigma = \mathcal{M} \to \rho$, for some $\mathcal{M}$ and ρ.

    Thus t is a weak error term with $\mathtt{F} = \Box$.
  - s is a weak structure with weak context $\mathtt{E}'$. We set $\mathtt{F} = \Box$ and conclude.
  - s is a weak error term with weak error context $\mathtt{F}'$. We set $\mathtt{F} = \texttt{case}\ \mathtt{F}'\ \texttt{of}\ \bar{b}$ and conclude.
- tES. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1 \oplus y :: \mathcal{M};\Sigma \vdash t_1 : \tau} \qquad \dfrac{\Pi_2}{\Gamma_2;\Sigma \vdash t_2 : \mathcal{M}}}{\Gamma_1\Gamma_2;\Sigma \vdash t_1[y\backslash t_2] : \tau}\ \text{tES}$$

Note that t_1 is π_1-normal. By the *i.h.* one of the following cases holds:

  - t_1 is an answer. Then so is t.
  - t_1 is a weak structure. We have two further cases to consider:
    - $t_1 = \mathtt{E}_1[x]$, with $x \neq y$. Then we set $\mathtt{E} := \mathtt{E}_1[y\backslash t_2]$ and conclude that t is a weak structure.
    - $t_1 = \mathtt{E}_1[y]$. From Lem. 33 applied to π_1, $\mathcal{M} \neq []$. From the *i.h.* on any derivation in Π_2, the fact that weak contexts are full contexts, and the fact that t is π-normal, t_2 must be a weak structure or weak error term. This concludes the case.
  - t_1 is a weak error term. Then $t_1 = \mathtt{F}_1[\mathit{case}...]$. Then we set $\mathtt{F} := \mathtt{F}_1[y\backslash t_2]$ and conclude that t is a weak error term.

$\Box$

Remark 35. $\mathbf{c} \notin \mathcal{P}(\tau)$ *implies* τ *is not a datatype. In other words,* τ *is of the form* α*,* $\mathcal{M} \to \sigma$ *or* E*.*

Recall that τ is good if $\mathbf{c} \notin \mathcal{P}(\tau)$ and $[] \notin \mathcal{N}(\tau)$. We say $\mathcal{M}$ is good if each $\tau \in \mathcal{M}$. Also, we say $\mathcal{M}$ is an error type when each $\tau \in \mathcal{M}$ is an error type.

Lemma 36 (Good/Error head variables in weak structures)**.** *Suppose* $\pi(\Gamma;\Sigma \vdash t : \tau)$*,* t *is a weak structure* $\mathtt{E}[x]$ *and* t *is* π*-normal.*

1. *If* $\Gamma(x)$ *is good, then* τ *is good.*
2. *If* $\Gamma(x)$ *is an error type, then* τ *is an error type*

Proof. We proceed by induction on π.

- TVAR. Then result is immediate.
- tABS, tCONS, tFIX, tCASE. The result is immediate since abstractions, constants, fixed point expressions and case expressions are not weak structures.
- tAPP. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1; \Sigma \vdash t_1 : \sigma} \qquad \sigma \mathbin{@} \mathcal{M} \Rightarrow \tau \qquad \Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}{\Gamma_1\Gamma_2; \Sigma \vdash t_1 t_2 : \tau}\ \text{tAPP}$$

  Then $\mathtt{E} = \mathtt{E}'\, t_2$ and $\mathtt{E}'[x] = t_1$. From the *i.h.* on π_1 we have two cases:

  - Suppose $\Gamma(x)$ is good. Then σ is good. From $\mathbf{c} \notin \mathcal{P}(\sigma)$ and Rem. 35, σ is either of the form (a) α; or (b) $\mathcal{M} \to \rho$; or (c) E. Case (a) is not possible since $\sigma \mathbin{@} \mathcal{M} \Rightarrow \tau$ would not be defined. From case (b) we deduce that $\tau = \rho$ and hence $\mathbf{c} \notin \mathcal{P}(\tau)$. For case (c), we deduce $\mathbf{c} \notin \mathcal{P}(\tau)$ from $\mathbf{c} \notin \mathcal{P}(\sigma)$ since $\tau \simeq \sigma$. The predicate $[] \notin \mathcal{N}(\tau)$ may be deduced similarly. Hence τ is good.
  - Suppose $\Gamma(x)$ is an error type. Then σ is an error type and hence so is τ.

- tES. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1 \oplus y :: \mathcal{M}; \Sigma \vdash t_1 : \tau} \qquad \dfrac{\Pi_2}{\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}}{\Gamma_1\Gamma_2; \Sigma \vdash t_1[y\backslash t_2] : \tau}\ \text{tES}$$

  We know that either:

  - $\mathtt{E} = \mathtt{E}_1[y\backslash t_2]$ with $x \neq y$; or
  - $\mathtt{E} = \mathtt{E}_1[\![y]\!][y\backslash \mathtt{E}_2]$;

  In the first case we apply the *i.h.* to π_1 and conclude. In the second item, first note that from Lem. 33 on π_1, $\mathcal{M} \neq []$. From the *i.h.* on each derivation in Π_2 (t_2 is also a weak structure), either $\mathcal{M}$ is good or an error type. We can then apply the *i.h.* on π_1 and conclude.

□

Γ is good if $\Gamma = \Gamma_g\Gamma_e$ and $\forall x \in \operatorname{dom}\Gamma_g$, $\Gamma_g(x)$ is good and $\forall x \in \operatorname{dom}\Gamma_e$, $\Gamma_e(x)$ is an error type.

Lemma 37 (Weak error terms have error types)**.** *Suppose* $\pi(\Gamma; \Sigma \vdash t : \tau)$*,* t *is a weak error term,* t *is* π*-normal and* Γ *is good, then* τ *is an error type.*

Proof. We proceed by induction on π.

- TVar, tAbs, tCons, and tFix. The result is immediate since variables, abstractions, constants and fixed point expressions are not weak error terms.

- tApp. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1; \Sigma \vdash t_1 : \sigma} \qquad \sigma \mathbin{@} \mathcal{M} \Rightarrow \tau \qquad \Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}{\Gamma_1\Gamma_2; \Sigma \vdash t_1 t_2 : \tau} \text{ tApp}$$

  Then $\mathtt{F} = \mathtt{F}'\, t_2$ and $\mathtt{F}'[case] = t_1$. From the *i.h.* on π_1 we know σ is an error type. Hence, so is τ.

- tCase. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1; \Sigma \vdash t_1 : \sigma} \qquad \sigma \ \langle \bar{b} \rangle \ \Gamma_2; \Sigma, \tau}{\Gamma_1\Gamma_2; \Sigma \vdash \texttt{case } t_1 \texttt{ of } \bar{b} : \tau} \text{ tCMatch}$$

  We have two cases:

  - $\mathtt{F} = \texttt{case } \mathtt{F}' \texttt{ of } \bar{b}$ and $\mathtt{F}'[case] = t_1$. From the *i.h.* on π_1 we deduce that σ is an error type. Hence so is τ.
  - $\mathtt{F} = \Box$. We have two cases:
    * t_1 is an answer. By Lem. 20,
      · If $t_1 = \mathtt{A}[\mathtt{c}]\mathtt{L}$, then $\sigma = \mathbf{c}\, \mathcal{M}_1 \ldots \mathcal{M}_n$ where $n = |\mathtt{A}|$. Since t_1 does not enable $\bar{b}$, then it must be the case that σ does not match $\bar{b}$. Then τ is an error type.
      · If $t_1 = (\lambda x.s)\mathtt{L}$, then $\sigma = \mathcal{M} \to \rho$. Same as above.
    * t_1 is a weak structure. By Lem. 36 σ is either good or an error type. In either case, τ is an error type.

- tES. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1 \oplus y :: \mathcal{M}; \Sigma \vdash t_1 : \tau} \qquad \dfrac{\Pi_2}{\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}}{\Gamma_1\Gamma_2; \Sigma \vdash t_1[y \backslash t_2] : \tau} \text{ tES}$$

  We have two cases

  - $\mathtt{F} = \mathtt{F}_1[y \backslash t_2]$. If $\mathcal{M} = []$, we conclude from the *i.h.* on π_1. Suppose that $\mathcal{M} \neq []$. By Lem. 34 and the fact that t is in π-normal form, t_2 must be a weak structure or weak error term and $y \in \mathsf{fv}(t_1)$. Indeed, if $y \notin \mathsf{fv}(t_1)$, then t would be a $\mathtt{gc}$ redex; if $y \in \mathsf{fv}(t_1)$ and t_2 were an answer, we would have a $\mathtt{lsv}$ redex. If t_2 is a weak structure, then by Lem. 36 $\mathcal{M}$ is either good or a multiset of error types. If t_2 is a weak error term, then by the *i.h.* $\mathcal{M}$ is an error type. In any case, we can apply the *i.h.* on π_1 and conclude.
  - $\mathtt{F} = \mathtt{E}_1[\![y]\!][y \backslash \mathtt{F}_2]$. From Lem. 33 on π_1 we know $\mathcal{M} \neq []$. We apply the *i.h.* on all derivations in Π_2 and deduce that $\mathcal{M}$ are error types. From Lem. 36 we conclude, that is, τ is an error type.

□

Lemma 38 (Coverage for weak structures)**.** *Suppose $\pi(\Gamma; \Sigma \vdash t : \tau)$, t is a weak structure $\mathtt{E}[x]$, t is π-normal and $\Gamma(x)$ is good or an error type and $\mathsf{covered}_\Sigma(\Gamma(x))$. Then $\mathsf{covered}_\Sigma(\tau)$.*

Proof. We proceed by induction on π.

- TVAR. Then result is immediate.
- TABS, TCONS, TFIX, TCASE. The result is immediate since abstractions, constants, fixed point expressions and case expressions are not weak structures.
- TAPP. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1; \Sigma \vdash t_1 : \sigma} \qquad \sigma \;@\; \mathcal{M} \Rightarrow \tau \qquad \Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}{\Gamma_1\Gamma_2; \Sigma \vdash t_1 t_2 : \tau}\;\textsc{tApp}$$

Then $\mathtt{E} = \mathtt{E}'\, t_2$ and $\mathtt{E}'[x] = t_1$. From the *i.h.* on π_1 we know that $\mathsf{covered}_\Sigma(\sigma)$. From Lem. 36 on π_1 we have two cases:

  - σ is good. From Rem. 35, σ is either of the form (a) α; or (b) $\mathcal{M} \to \rho$; or (c) E. Case (a) is not possible since $\sigma \;@\; \mathcal{M} \Rightarrow \tau$ would not be defined. From case (b) we deduce that $\tau = \rho$ and hence $\mathsf{covered}_\Sigma(\tau)$ follows from $\mathsf{covered}_\Sigma(\sigma)$. For case (c), since $\tau \simeq \sigma$, then again $\mathsf{covered}_\Sigma(\tau)$ follows from $\mathsf{covered}_\Sigma(\sigma)$.
  - σ is an error type. Then $\tau \simeq \sigma$ and hence $\mathsf{covered}_\Sigma(\tau)$ follows from $\mathsf{covered}_\Sigma(\sigma)$.

- TES. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1 \oplus y :: \mathcal{M}; \Sigma \vdash t_1 : \tau} \qquad \dfrac{\Pi_2}{\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}}{\Gamma_1\Gamma_2; \Sigma \vdash t_1[y\backslash t_2] : \tau}\;\textsc{tES}$$

We know that either:

  - $\mathtt{E} = \mathtt{E}_1[y\backslash t_2]$ with $x \neq y$; or
  - $\mathtt{E} = \mathtt{E}_1[\![y]\!][y\backslash \mathtt{E}_2]$;

In the first case we apply the *i.h.* to π_1 and conclude. In the second item, first note that from Lem. 33 on π_1, $\mathcal{M} \neq [\,]$. From Lem. 36 on any derivation in Π_2, (t_2 is also a weak structure), $\mathcal{M}$ is either good or an error type. Also, from the *i.h.* on each derivation in Π_2, $\mathsf{covered}_\Sigma(\mathcal{M})$. We can then apply the *i.h.* on π_1 and conclude.

□

Lemma 39 (Coverage for weak error terms)**.** *Suppose $\pi(\Gamma; \Sigma \vdash t : \tau)$, t is a weak error term, t is π-normal and Γ is good and $\mathsf{covered}_\Sigma(\Gamma)$. Then $\mathsf{covered}_\Sigma(\tau)$.*

Proof. We proceed by induction on π.

- TVAR, TABS, TCONS, TFIX. The result is immediate since variables, abstractions, constants and fixed point expressions are not weak error terms.
- TAPP. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1; \Sigma \vdash t_1 : \sigma} \qquad \sigma \mathbin{@} \mathcal{M} \Rightarrow \tau \qquad \Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}{\Gamma_1\Gamma_2; \Sigma \vdash t_1 t_2 : \tau} \text{ TAPP}$$

Then $\mathtt{F} = \mathtt{F}'\, t_2$ and $\mathtt{F}'[case..] = t_1$. From the *i.h.* on π_1 we know that $\mathsf{covered}_\Sigma(\sigma)$. From Lem. 37 σ is an error type. Then $\tau \simeq \sigma$ and hence $\mathsf{covered}_\Sigma(\tau)$.

- TCASE. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1; \Sigma \vdash t_1 : \sigma} \qquad \sigma \ \langle \bar{b} \rangle \ \Gamma_2; \Sigma, \tau}{\Gamma_1\Gamma_2; \Sigma \vdash \mathtt{case}\ t_1\ \mathtt{of}\ \bar{b} : \tau} \text{ TCMATCH}$$

We have two cases:

  - $\mathtt{F} = \mathtt{case}\ \mathtt{F}'\ \mathtt{of}\ \bar{b}$ and $\mathtt{F}'[case] = t_1$. From Lem. 37 σ is an error type. Then $\tau \in \Sigma$ and hence $\mathsf{covered}_\Sigma(\tau)$.
  - $\mathtt{F} = \Box$. There are two cases:
    * t_1 is an answer. By Lem. 20,
      - If $t_1 = \mathtt{A}[\mathbf{c}]\mathtt{L}$, then $\sigma = \mathbf{c}\, \mathcal{M}_1 \ldots \mathcal{M}_n$ where $n = |\mathtt{A}|$. Since t_1 does not enable $\bar{b}$, then it must be the case that σ does not match $\bar{b}$. Same as above.
      - If $t_1 = (\lambda x.s)\mathtt{L}$, then $\sigma = \mathcal{M} \to \rho$. Same as above.
    * t_1 is a weak structure. From Lem. 36 σ is either good or an error type. In any case, σ does not match $\bar{b}$. Then $\tau \in \Sigma$ and hence $\mathsf{covered}_\Sigma(\tau)$.

- TES. The derivation ends in:

$$\dfrac{\dfrac{\pi_1}{\Gamma_1 \oplus y :: \mathcal{M}; \Sigma \vdash t_1 : \tau} \qquad \dfrac{\Pi_2}{\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}}{\Gamma_1\Gamma_2; \Sigma \vdash t_1[y \backslash t_2] : \tau} \text{ TES}$$

We have two cases:

  - $\mathtt{F} = \mathtt{F}_1[y \backslash t_2]$. Then $y \in \mathsf{fv}(t_1)$ and t_2 is a weak structure or a weak error term. Indeed, if $y \notin \mathsf{fv}(t_1)$, then t would be a $\mathtt{gc}$ redex; if $y \in \mathsf{fv}(t_1)$, and t_2 were an answer, we would have a $\mathtt{lsv}$ redex. If t_2 is a weak structure, then by Lem. 36 $\mathcal{M}$ is either good or a multiset of error types. Moreover, by Lem. 38, $\mathsf{covered}_\Sigma(\mathcal{M})$. If t_2 is a weak error term, then by Lem. 37 $\mathcal{M}$ is an error type. From the *i.h.* on any derivation in Π_2, $\mathsf{covered}_\Sigma(\mathcal{M})$. Hence $\mathsf{covered}_\Sigma(\Gamma_1 \oplus y :: \mathcal{M})$. In any case, we can apply the *i.h.* on π_1 and conclude.
  - $\mathtt{F} = \mathtt{E}_1[\![y]\!][y \backslash \mathtt{F}_2]$. From Lem. 33 applied to π_1, $\mathcal{M} \neq []$. By the *i.h.* on Π_2, $\mathsf{covered}_\Sigma(\mathcal{M})$. By Lem. 37 $\mathcal{M}$ is an error type. We then conclude from Lem. 38 on π_1.

$\square$

We make use of the following weakened notion of good. A judgement $\Gamma; \Sigma \vdash t : \tau$ is said to be **good-minus** if:

- Γ is good (*i.e.* $\Gamma = \Gamma_g \Gamma_e$ and $\forall x \in \operatorname{dom} \Gamma_g$, $\Gamma_g(x)$ is good and $\forall x \in \operatorname{dom} \Gamma_e$, $\Gamma_e(x)$ is an error type).
- $[] \notin \mathcal{P}(\Sigma)$.
- $\mathbf{c} \notin \mathcal{N}(\Sigma)$.
- $\mathsf{covered}_\Sigma(\Gamma)$.

In order to compare it with the previous notion of a good judgement, we recall its definition below: A judgement $\Gamma; \Sigma \vdash t : \tau$ is said to be **good** if

- Γ is good (*i.e.* $\Gamma = \Gamma_g \Gamma_e$ and $\forall x \in \operatorname{dom} \Gamma_g$, $\Gamma_g(x)$ is good and $\forall x \in \operatorname{dom} \Gamma_e$, $\Gamma_e(x)$ is an error type).
- $[] \notin \mathcal{P}(\Sigma, \tau)$.
- $\mathbf{c} \notin \mathcal{N}(\Sigma, \tau)$ and
- $\mathsf{covered}_\Sigma(\Gamma)$ and $\mathsf{covered}_\Sigma(\tau)$.

Lemma 40. *Suppose $\pi(\Gamma; \Sigma \vdash t : \tau)$, t is π-normal.*

1. *If t is an abstraction answer and $\Gamma; \Sigma \vdash \tau$ is good, then $t \in \mathcal{L}$.*
2. *If t is a constant answer and $\Gamma; \Sigma \vdash \tau$ is good, then $t \in \mathcal{K}$.*
3. *If t is a weak structure and $\Gamma; \Sigma \vdash \tau$ is good-minus, then $t \in \mathcal{S}$.*
4. *If t is a weak error term and $\Gamma; \Sigma \vdash \tau$ is good-minus, then $t \in \mathcal{E}$.*
5. *Moreover, if $x \in \mathsf{fv}(t)$, then x has some typed occurrence in π.*

Proof. By induction on π.

- $\textsc{tVar}$. Then $t = y$ and t is a weak structure. Clearly $y \in \mathcal{S}$. Moreover, y has a typed occurrence in π.
- $\textsc{tAbs}$. Then $t = \lambda x.s$ and the derivation π ends as follows:

$$\dfrac{\dfrac{\pi_1}{\Gamma \oplus x :: \mathcal{M}; \Sigma \vdash s : \sigma}}{\Gamma; \Sigma \vdash \lambda x.s : \mathcal{M} \to \sigma}\ \textsc{tAbs}$$

  Clearly, s must be π_1-normal.

  Claim: $\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash \sigma$ is good. $\Gamma \oplus x :: \mathcal{M}$ good follows from Γ good and $\mathcal{M}$ good; the latter follows from $[] \notin \mathcal{P}(\mathcal{M} \to \sigma)$ and $\mathbf{c} \notin \mathcal{N}(\mathcal{M} \to \sigma)$. $[] \notin \mathcal{P}(\Sigma, \sigma)$ follows from

$[] \notin \mathcal{P}(\Sigma, \mathcal{M} \to \sigma)$. Likewise, $\mathbf{c} \notin \mathcal{N}(\Sigma, \sigma)$ follows from $\mathbf{c} \notin \mathcal{N}(\Sigma, \mathcal{M} \to \sigma)$. Finally, $\mathsf{covered}_\Sigma(\Gamma \oplus x : \mathcal{M}, \sigma)$ follows from $\mathsf{covered}_\Sigma(\Gamma, \mathcal{M} \to \sigma)$.

Thus by the *i.h.* and nfCons, nfLam, nfStruct, nfError, depending on whether s is an answer, a weak structure or weak error term $s \in \mathcal{N}$ and we conclude using lNfLam.

If $y \in \mathsf{fv}(t)$, then $y \in \mathsf{fv}(s \setminus \{x\})$ and we may conclude that y has some typed occurrence in π from the *i.h.*.

- tCons. Then $\Gamma = \varnothing$ and $t = \mathbf{c}$ and clearly $\mathbf{c} \in \mathcal{K}$. Note that $\mathbf{c} \notin \mathcal{N}(\mathbf{c})$.
- tApp. Then $t = t_1\, t_2$ and the derivation π ends in

$$\dfrac{\dfrac{\pi_1}{\Gamma_1; \Sigma \vdash t_1 : \sigma} \qquad \dfrac{\pi_2}{\sigma \mathbin{@} \mathcal{M} \Rightarrow \tau} \qquad \dfrac{\Pi_3}{\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}}{\Gamma_1 + \Gamma_2; \Sigma \vdash t_1\, t_2 : \tau}\ \text{tApp}$$

where $\Gamma = \Gamma_1 + \Gamma_2$. Since, t is not an abstraction answer, by Lem. 34, we consider two cases:

  - t is a constant answer. Note that t_1 must be a constant answer too and t_1 is π_1-normal since t is π-normal. From Lem. 20(1) on π_1, $\sigma = \mathbf{c}\,\mathcal{M}_1 \ldots \mathcal{M}_n$. Therefore, $\tau = \mathbf{c}\,\mathcal{M}_1 \ldots \mathcal{M}_n\,\mathcal{M}$.

    Claim: $\Gamma_1; \Sigma \vdash \sigma$ *is good.* $[] \notin \mathcal{P}(\Sigma, \tau)$ implies $[] \notin \mathcal{P}(\Sigma, \sigma)$; likewise $\mathbf{c} \notin \mathcal{N}(\Sigma, \tau)$ implies $\mathbf{c} \notin \mathcal{N}(\Sigma, \sigma)$. Γ_1 good follows from Γ good. Finally, from $\mathsf{covered}_\Sigma(\Gamma, \tau)$ we know $\mathsf{covered}_\Sigma(\Gamma_1, \sigma)$.

    We may thus apply the *i.h.* on π_1 to obtain $t_1 \in \mathcal{K}$.

    Claim: $\Gamma_2; \Sigma \vdash \mathcal{M}$ *is good.* $[] \notin \mathcal{P}(\Sigma, \tau)$ also implies $\mathcal{M} \neq []$. That $\Gamma_2; \Sigma \vdash \rho$ good, for all $\rho \in \mathcal{M}$, may be proved as above.

    This allows us to apply the *i.h.* to any derivation in Π_3 and deduce $t_2 \in \mathcal{N}$. We conclude $t \in \mathcal{S}$ from rule sNfApp.

    Note that $x \in \mathsf{fv}(t)$ implies $x \in \mathsf{fv}(t_1)$ or $x \in \mathsf{fv}(t_2)$. For item (5) we conclude from the *i.h.* on π_1 or π_2 in Π_3.
  - t is a weak structure. Then t_1 is also a weak structure and Γ_1 is good. From Lem. 36 on π_1, we have two cases:
    * σ is good:

$$\mathbf{c} \notin \mathcal{P}(\sigma) \tag{11}$$
$$[] \notin \mathcal{N}(\sigma) \tag{12}$$

      Claim: $\Gamma_2; \Sigma \vdash \mathcal{M}$ *is good.* From (11) and Rem. 35, σ is of one of the following forms: α, $\mathcal{M} \to \tau$, or E. The first case is not possible since then $\sigma \mathbin{@} \mathcal{M} \Rightarrow \tau$ would not be defined. Suppose $\sigma = \mathcal{M} \to \tau$. From (12) we deduce $[] \notin \mathcal{P}(\mathcal{M})$. From (11) we deduce $\mathbf{c} \notin \mathcal{N}(\mathcal{M})$. Hence $[] \notin \mathcal{P}(\Sigma, \mathcal{M})$ and $\mathbf{c} \notin \mathcal{N}(\Sigma, \mathcal{M})$, since by hypothesis we already know that $[] \notin \mathcal{P}(\Sigma)$ and $\mathbf{c} \notin \mathcal{N}(\Sigma)$. By Lem. 38 on π_1 we also know that $\mathsf{covered}_\Sigma(\mathcal{M} \to \tau)$; hence also $\mathsf{covered}_\Sigma(\mathcal{M})$. Γ_2 good follows from Γ good. This proves the claim that $\Gamma_2; \Sigma \vdash \mathcal{M}$ is good.

      This allows us to apply the *i.h.* to any derivation in Π_3 and deduce $t_2 \in \mathcal{N}$.

* σ is an error type $\langle \mathbb{e}\, \tau\, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i \in I}\, \rho_1 \ldots \rho_j \rangle\, \rho_{j+1} \ldots \rho_k$ and $\mathcal{M} = [\rho_{j+1}]$. Then $\sigma \simeq \tau$.
  Claim: $\Gamma_2; \Sigma \vdash \mathcal{M}$ *is good.* By Lem. 38, $\mathsf{covered}_\Sigma(\sigma)$ and hence also $\mathsf{covered}_\Sigma(\rho_{j+1})$. From $[] \notin \mathcal{P}(\Sigma)$ and $\mathbf{c} \notin \mathcal{N}(\Sigma)$ we deduce $[] \notin \mathcal{P}(\rho_{j+1})$ and $\mathbf{c} \notin \mathcal{N}(\rho_{j+1})$. Thus $[] \notin \mathcal{P}(\Sigma, \rho_{j+1})$ and $\mathbf{c} \notin \mathcal{N}(\Sigma, \rho_{j+1})$. Γ_2 good follows from Γ good.
  This allows us to apply the *i.h.* to the only derivation in Π_3 and deduce $t_2 \in \mathcal{N}$.

Since t_1 is a weak structure we now prove that $\Gamma_1; \Sigma \vdash t_1 : \sigma$ is good-minus.

Claim: $\Gamma_1; \Sigma \vdash t_1 : \sigma$ *is good-minus.* Γ_1 good follows from Γ good. $\mathsf{covered}_\Sigma(\Gamma_1)$ follows from $\mathsf{covered}_\Sigma(\Gamma)$. $[] \notin \mathcal{P}(\Sigma)$ and $\mathbf{c} \notin \mathcal{N}(\Sigma)$ follows from the same assumptions.

From the *i.h.* on π_1, $t_1 \in \mathcal{S}$. Using sNfApp or eNfApp concludes the case.

Suppose now that $x \in \mathsf{fv}(t)$. Then either $x \in \mathsf{fv}(t_1)$ or $x \in \mathsf{fv}(t_2)$. We conclude that x occurs typed in π from either the *i.h.* on π_1 or any derivation in Π_3.

– t is a weak error term. *Claim:* $\Gamma_2; \Sigma \vdash \mathcal{M}$ *is good.* Since t_1 is also a weak error term and Γ_1 is good, from Lem. 37 on π_1, σ is an error type $\langle \mathbb{e}\, \tau\, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i \in I}\, \rho_1 \ldots \rho_j \rangle\, \rho_{j+1} \ldots \rho_k$ and $\mathcal{M} = [\rho_{j+1}]$. From Lem. 39, $\mathsf{covered}_\Sigma(\sigma)$. $\mathbf{c} \notin \mathcal{N}(\Sigma)$ and $[] \notin \mathcal{P}(\Sigma)$ follow from the same assumptions. $\mathbf{c} \notin \mathcal{N}(\tau)$ and $[] \notin \mathcal{P}(\tau)$ follow from $\mathsf{covered}_\Sigma(\sigma)$ and $\mathbf{c} \notin \mathcal{N}(\Sigma)$ and $[] \notin \mathcal{P}(\Sigma)$.

  This allows us to apply the *i.h.* to the only derivation in Π_3 and deduce $t_2 \in \mathcal{N}$.

  Finally, since t_1 is also a weak error term and $\Gamma_1; \Sigma \vdash t_1 : \sigma$ is good-minus, from the *i.h.* on π_1, $t_1 \in \mathcal{E}$. Using eNfApp concludes the case.

  Suppose now that $x \in \mathsf{fv}(t)$. Then either $x \in \mathsf{fv}(t_1)$ or $x \in \mathsf{fv}(t_2)$. We conclude that x occurs typed in π from either the *i.h.* on π_1 or any derivation in Π_3.

- tFix. This case is not possible since otherwise t would not be π-normal.
- tCase. The derivation ends as follows, where $\bar{b}$ abbreviates $(\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I}$.

$$\dfrac{\dfrac{\pi_1}{\Gamma_1; \Sigma \vdash s : \sigma} \qquad \dfrac{\pi_2}{\sigma\ \langle \bar{b} \rangle\ \Gamma_2; \Sigma, \tau}}{\Gamma_1 + \Gamma_2; \Sigma \vdash \texttt{case}\ s\ \texttt{of}\ \bar{b} : \tau}\ \textsc{tCMatch}$$

Since t is not an answer and is a case expression, by Lem. 34, t is a weak error term. We consider three cases.

– s is an answer. From Lem. 20 on π_1, either $\sigma = \mathcal{M} \to \rho$ or $\sigma = \mathbf{c}\, \mathcal{M}_1 \ldots \mathcal{M}_n$. Neither of these types match $\bar{b}$ and, therefore, from $\sigma\ \langle \bar{b} \rangle\ \Gamma_2; \Sigma_2, \tau$ we deduce
  * $\tau = \langle \mathbb{e}\, \sigma\, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i \in I} \rangle\, \rho_1 \ldots \rho_k$ and
  * $\pi_{2i}(\Gamma_{2i} \oplus \bar{x}_i :: \bar{\mathcal{M}}_i; \Sigma \vdash s_i : \sigma_i)_{i \in I}$, with $\Gamma_2 = \sum_{i \in I} \Gamma_{2i}$ and $\tau \in \Sigma$.

  Claim: $\Gamma_1; \Sigma \vdash s : \sigma$ *is good.* Since $\mathsf{covered}_\Sigma(\tau)$, also $\mathsf{covered}_\Sigma(\sigma)$. Since $[] \notin \mathcal{P}(\Sigma)$, then $[] \notin \mathcal{P}(\sigma)$. Similarly, since $\mathbf{c} \notin \mathcal{N}(\Sigma)$, then $\mathbf{c} \notin \mathcal{N}(\sigma)$. Hence $[] \notin \mathcal{P}(\Sigma, \sigma)$ and $\mathbf{c} \notin \mathcal{N}(\Sigma, \sigma)$. Finally, Γ_1 good follows from Γ good. Therefore $\Gamma_1; \Sigma \vdash \sigma$ is good.

  Moreover, since t is π-normal, s is π_1-normal and we can apply the *i.h.* obtaining $s \in \mathcal{L} \cup \mathcal{K}$.

  Claim: $\Gamma_{2i} \oplus \bar{x}_i :: \bar{\mathcal{M}}_i; \Sigma \vdash s_i : \sigma_i$, *is good for each* $i \in I$. Indeed, $[] \notin \mathcal{N}(\bar{\mathcal{M}}_i)$ and $\mathbf{c} \notin \mathcal{P}(\bar{\mathcal{M}}_i)$ and $\mathbf{c} \notin \mathcal{N}(\sigma_i)$ and $[] \notin \mathcal{P}(\sigma_i)$ follow from $\mathsf{covered}_\Sigma(\sigma)$ and $[] \notin \mathcal{P}(\Sigma)$ and

$\mathbf{c} \notin \mathcal{N}(\Sigma)$. Hence $\bar{\mathcal{M}}_i$ is good for each $i \in I$. Also, $\mathsf{covered}_\Sigma(\tau)$ implies $\mathsf{covered}_\Sigma(\bar{\mathcal{M}}_i, \sigma_i)$. Finally, Γ_{2i} good follows from Γ good.

Also, t is π-normal implies s_i are π_{2i}-normal. We can apply the *i.h.* obtaining $s_i \in \mathcal{N}$. We conclude from eNfStrt.

– s is a weak structure. Since Γ_1 is good, then from Lem. 36 on π_1, we have two cases:

  * σ is good:

$$\mathbf{c} \notin \mathcal{P}(\sigma) \tag{13}$$

$$[] \notin \mathcal{N}(\sigma) \tag{14}$$

    Claim: $\Gamma_{2i} \oplus \bar{x}_i :: \bar{\mathcal{M}}_i; \Sigma \vdash s_i : \sigma_i$*, is good for each* $i \in I$. From (11) and Rem. 35, σ is of one of the following forms: α, $\mathcal{M} \to \tau$ or E. The first case is not possible since then $\sigma \mathbin{@} \mathcal{M} \Rightarrow \tau$ would not be defined. Suppose $\sigma = \mathcal{M} \to \tau$. From (14) we deduce $[] \notin \mathcal{P}(\mathcal{M})$. From (13) we deduce $\mathbf{c} \notin \mathcal{N}(\mathcal{M})$. Hence $[] \notin \mathcal{P}(\Sigma, \mathcal{M})$ and $\mathbf{c} \notin \mathcal{N}(\Sigma, \mathcal{M})$, since by hypothesis we already know that $[] \notin \mathcal{P}(\Sigma)$ and $\mathbf{c} \notin \mathcal{N}(\Sigma)$. By Lem. 38 on π_1 we also know that $\mathsf{covered}_\Sigma(\mathcal{M} \to \tau)$; hence also $\mathsf{covered}_\Sigma(\mathcal{M})$. The remaining items in order to prove that $\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}$ is good is addressed as in the case where t_1 is a constant answer. This allows us to apply the *i.h.* to any derivation in Π_3 and deduce $t_2 \in \mathcal{N}$.

  * σ is an error type $\langle \mathbb{e}\, \tau\, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i \in I}\, \rho_1 \ldots \rho_j \rangle\, \rho_{j+1} \ldots \rho_k$ and $\mathcal{M} = [\rho_{j+1}]$. Then $\sigma \simeq \tau$.
    Claim: $\Gamma_{2i} \oplus \bar{x}_i :: \bar{\mathcal{M}}_i; \Sigma \vdash s_i : \sigma_i$*, is good for each* $i \in I$. By Lem. 38, $\mathsf{covered}_\Sigma(\sigma)$. From $[] \notin \mathcal{P}(\Sigma)$ and $\mathbf{c} \notin \mathcal{N}(\Sigma)$ we deduce $[] \notin \mathcal{P}(\rho_{j+1})$ and $\mathbf{c} \notin \mathcal{N}(\rho_{j+1})$. Thus $[] \notin \mathcal{P}(\Sigma, \rho_{j+1})$ and $\mathbf{c} \notin \mathcal{N}(\Sigma, \rho_{j+1})$. Γ_2 good follows from Γ good. Finally, from $\mathsf{covered}_\Sigma(\sigma)$ we know that also $\mathsf{covered}_\Sigma(\rho_{j+1})$.
    This allows us to apply the *i.h.* to the only derivation in Π_3 and deduce $t_2 \in \mathcal{N}$.

  Moreover, $\Gamma_1; \Sigma \vdash t_1 : \sigma$ can be shown to be good. From the *i.h.* on π_1, $t_1 \in \mathcal{S}$. Using sNfApp or eNfApp concludes the case.

  Suppose now that $x \in \mathsf{fv}(t)$. Then either $x \in \mathsf{fv}(s)$ or $x \in \mathsf{fv}(\bar{b})$. We conclude that x occurs typed in π from either the *i.h.* on π_1 or any derivation in Π_3.

– s is a weak error term.
  Claim: $\Gamma_{2i} \oplus \bar{x}_i :: \bar{\mathcal{M}}_i; \Sigma \vdash s_i : \sigma_i$*, is good for each* $i \in I$. Since Γ_1 is good, from Lem. 37 on π_1, σ is an error type $\langle \mathbb{e}\, \tau\, (\bar{\mathcal{M}}_i \Rightarrow \sigma_i)_{i \in I}\, \rho_1 \ldots \rho_j \rangle\, \rho_{j+1} \ldots \rho_k \in \Sigma$. From Lem. 39, $\mathsf{covered}_\Sigma(\sigma)$. From the $\mathbf{c} \notin \mathcal{N}(\Sigma)$ we deduce $\mathbf{c} \notin \mathcal{N}(\rho_{j+1})$. From the $[] \notin \mathcal{P}(\Sigma)$ we deduce $[] \notin \mathcal{P}(\rho_{j+1})$. This allows us to apply the *i.h.* to the only derivation in Π_3 and deduce $t_2 \in \mathcal{N}$.

  Finally, since t_1 is also a weak error term and $\Gamma_1; \Sigma \vdash t_1 : \sigma$ is good, from the *i.h.* on π_1, $t_1 \in \mathcal{E}$. Using eNfApp concludes the case.

• tES.

$$\dfrac{\dfrac{\pi_1}{\Gamma_1 \oplus y :: \mathcal{M}; \Sigma \vdash t_1 : \tau} \qquad \dfrac{\Pi_2}{\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}}}{\Gamma_1 + \Gamma_2; \Sigma \vdash t_1[y \backslash t_2] : \tau}\ \text{tES}$$

Since t is in π-normal form, $y \in \mathsf{fv}(t_1)$ and t_2 cannot be an answer. Note that $\mathcal{M} \neq []$: if $\mathcal{M} = []$, then $\Gamma_1; \Sigma \vdash \tau$ good-minus follows from $\Gamma; \Sigma \vdash \tau$ good [-minus] and we could apply the *i.h.* w.r.t item (5) and deduce that y has a typed occurrence in π_1, in which case $\mathcal{M} \neq []$. By Lem. 34 on any derivation in Π_2 and the fact that t_2 is not an answer, t_2 must be a weak structure or a weak error term.

We prove $\Gamma_2; \Sigma \vdash t_2 : \mathcal{M}$ is good-minus. Γ_2 good follows from Γ good. $\mathsf{covered}_\Sigma(\Gamma_2)$ follows from $\mathsf{covered}_\Sigma(\Gamma)$. Finally, $[] \notin \mathcal{P}(\Sigma)$ and $\mathbf{c} \notin \mathcal{N}(\Sigma)$ are immediate from the same hypothesis. From the *i.h.* on any derivation in Π_2 we have

$$t_2 \in \mathcal{N} \tag{15}$$

We next verify that $\Gamma_1 \oplus y :: \mathcal{M}$ is good.

- t_2 is a weak structure. Since Γ_2 is good, by Lem. 36, $\mathcal{M}$ is good or is an error type. Then $\Gamma_1 \oplus y :: \mathcal{M}$ is good.
- t_2 is a weak error term. By Lem. 37 $\mathcal{M}$ is an error type. Γ_1 good follows from Γ good. Thus we know that $\Gamma_1 \oplus x :: \mathcal{M}$ is good.

We next consider various cases:

- t is an abstraction answer. *Claim:* $\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash \tau$ *is good.* $[] \notin \mathcal{P}(\Sigma, \tau)$ and $\mathbf{c} \notin \mathcal{N}(\Sigma, \tau)$ are immediate from the same hypothesis. Lem. 38 and Lem. 39 on Π_2 allow us to deduce that $\mathsf{covered}_\Sigma(\mathcal{M})$. $\mathsf{covered}_\Sigma(\Gamma_1)$ follows from $\mathsf{covered}_\Sigma(\Gamma)$. Thus $\mathsf{covered}_\Sigma(\Gamma_1 \oplus y :: \mathcal{M})$.

  The *i.h.* on π_1 gives us $t_1 \in \mathcal{L}$. We conclude from this, (15) and NFSUB.
- t is a constant answer. *Claim:* $\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash \tau$ *is good.* As above. The *i.h.* on π_1 gives us $t_1 \in \mathcal{K}$. We conclude from this, (15) and NFSUB.
- t is a weak structure. *Claim:* $\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash \tau$ *is good-minus.* As above. The *i.h.* on π_1 gives us $t_1 \in \mathcal{S}$. We conclude from this, (15) and NFSUB.
- t is a weak error term. *Claim:* $\Gamma_1 \oplus x :: \mathcal{M}; \Sigma \vdash \tau$ *is good-minus.* As above. The *i.h.* on π_1 gives us $t_1 \in \mathcal{E}$. We conclude from this, (15) and NFSUB.

Suppose $x \in \mathsf{fv}(t)$. Then either $x \in \mathsf{fv}(t_1)$ or $x \in \mathsf{fv}(t_2)$. In either case we use the *i.h.*.

□

Lemma 41. *If $\pi(\Gamma; \Sigma \vdash t : \tau)$, t is in π-normal form and $\Gamma; \Sigma \vdash t : \tau$ is good, then $t \in \mathsf{NF}(\rightarrow_{\mathtt{sh}})$.*

Proof. Consequence of Lem. 40 and Lem. 13. □

Theorem 42 (Step 2)**.** *If $\pi(\Gamma; \Sigma \vdash t : \tau)$ and $\Gamma; \Sigma \vdash t : \tau$ is good, then t is definable in $\lambda_{\mathtt{sh}}$.*

Proof. By induction on the size of π. First, from Lem. 41, if $t \notin \mathsf{NF}(\rightarrow_{\mathtt{sh}})$, then t must have a typed redex occurrence in π. Let $t \rightarrow_{\mathtt{sh}} s$. From Lem. 31 there exists ξ such that $\xi(\Gamma; \Sigma \vdash s)$ and $\mathtt{M}(\xi) < \mathtt{M}(\pi)$. We conclude from the *i.h.* on s. □

Assembling Step 1 and Step 2 we obtain:

Figure 5 Evaluation Contexts

$$\frac{}{\Box \in \mathcal{C}^{\cdot}_{\vartheta}}\ \text{eBox}$$

$$\frac{\mathtt{C} \in \mathcal{C}^{h}_{\vartheta} \quad h \neq \lambda}{\mathtt{C}\, t \in \mathcal{C}^{h}_{\vartheta}}\ \text{eAppL} \qquad \frac{t \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta} \quad \mathtt{C} \in \mathcal{C}^{h}_{\vartheta}}{t\, \mathtt{C} \in \mathcal{C}^{\cdot}_{\vartheta}}\ \text{eAppRStruct} \qquad \frac{t \in \mathcal{K}_{\vartheta} \quad \mathtt{C} \in \mathcal{C}^{h}_{\vartheta}}{t\, \mathtt{C} \in \mathcal{C}^{\mathsf{hc}(t)}_{\vartheta}}\ \text{eAppRCons}$$

$$\frac{\mathtt{C} \in \mathcal{C}^{h}_{\vartheta} \quad t \notin \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta} \quad x \notin \vartheta}{\mathtt{C}[x\backslash t] \in \mathcal{C}^{h}_{\vartheta}}\ \text{eSubsLNonStruct} \qquad \frac{\mathtt{C} \in \mathcal{C}^{h}_{\vartheta \cup \{x\}} \quad t \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}}{\mathtt{C}[x\backslash t] \in \mathcal{C}^{h}_{\vartheta}}\ \text{eSubsLStruct}$$

$$\frac{\mathtt{C}_1 \in \mathcal{C}^{h}_{\vartheta} \quad \mathtt{C}_2 \in \mathcal{C}^{\cdot}_{\vartheta}}{\mathtt{C}_1[\![x]\!][x\backslash \mathtt{C}_2] \in \mathcal{C}^{h}_{\vartheta}}\ \text{eSubsR} \qquad \frac{\mathtt{C} \in \mathcal{C}^{h}_{\vartheta \cup \{x\}}}{\lambda x.\mathtt{C} \in \mathcal{C}^{\lambda}_{\vartheta}}\ \text{eLam}$$

$$\frac{\mathtt{C} \in \mathcal{C}^{h}_{\vartheta} \quad h \notin \{\mathbf{c}_i\}_{i \in I} \text{ or } h = \mathbf{c}_j \in \{\mathbf{c}_i\}_{i \in I} \text{ and } |\mathtt{C}[y]| \neq |\bar{x_j}|}{\mathtt{case}\ \mathtt{C}\ \mathtt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \in \mathcal{C}^{\cdot}_{\vartheta}}\ \text{eCase1}$$

$$\frac{t \in \mathcal{N}_{\vartheta} \quad t \not\succ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \quad t_k \in \mathcal{N}_{\vartheta \cup \bar{x_k}} \text{ for all } k < j \quad \mathtt{C} \in \mathcal{C}^{h}_{\vartheta \cup \bar{x_i}}}{\mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1 \bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j \bar{x_j} \Rightarrow \mathtt{C}, \ldots, \mathbf{c}_n \bar{x_n} \Rightarrow t_n \in \mathcal{C}^{\cdot}_{\vartheta}}\ \text{eCase2}$$

Theorem 43 (Soundness of $\lambda_{\mathtt{sh}}$ w.r.t. $\lambda_{\mathtt{e}}$)**.** *Let t be a term in $\Lambda_{\mathtt{e}}$. If $t \in \mathtt{WN}(\to_{\mathtt{e}})$, then $t \in \mathtt{WN}(\to_{\mathtt{sh}})$. More precisely, if $t \twoheadrightarrow_{\mathtt{e}} \mathtt{n}_{\mathtt{e}}$, where $\mathtt{n}_{\mathtt{e}} \in \Lambda_{\mathtt{sh}}$ is a $\to_{\mathtt{e}}$-normal form, then $t \twoheadrightarrow_{\mathtt{sh}} \mathtt{n}_{\mathtt{sh}}$, where $\mathtt{n}_{\mathtt{sh}}$ is a $\to_{\mathtt{sh}}$-normal form. Moreover, $\mathtt{n}^{\diamond}_{\mathtt{sh}} = \mathtt{n}_{\mathtt{e}}$.*

Proof. Let $t \twoheadrightarrow_{\mathtt{e}} \mathtt{n}_{\mathtt{e}}$, where $\mathtt{n}_{\mathtt{e}}$ is in $\to_{\mathtt{e}}$-nf. Then $\pi(\Gamma; \Sigma \vdash t : \tau)$ and $\Gamma; \Sigma \vdash \tau$ is good by Thm. 25. But then t is weakly $\to_{\mathtt{sh}}$-normalising by Thm. 42, so that $t \twoheadrightarrow_{\mathtt{sh}} \mathtt{n}_{\mathtt{sh}}$, where $\mathtt{n}_{\mathtt{sh}}$ is in $\to_{\mathtt{sh}}$-nf. By Lem. 14(1) $t^{\diamond} \twoheadrightarrow_{\beta} \mathtt{n}^{\diamond}_{\mathtt{sh}}$ and by Lem. 14(2) $\mathtt{n}^{\diamond}_{\mathtt{sh}} \in \mathsf{NF}(\to_{\mathtt{e}})$. Since $t^{\diamond} = t \twoheadrightarrow_{\beta} \mathtt{n}_{\mathtt{e}}$ and $t^{\diamond} \twoheadrightarrow_{\mathtt{e}} \mathtt{n}^{\diamond}_{\mathtt{sh}}$, then we conclude $\mathtt{n}^{\diamond}_{\mathtt{sh}} = \mathtt{n}_{\mathtt{e}}$ because $\to_{\mathtt{e}}$ is Church-Rosser. ☐

5 The Strong Call-by-Need Strategy

The strong call-by-need *strategy* $\rightarrowtail^{\vartheta}_{\mathtt{sh}}$ is a binary relation over terms in $\Lambda_{\mathtt{sh}}$ and is parameterized over a set ϑ of variables called *frozen variables*. It is defined by means of reduction rules similar to those given for the theory of sharing (Def. 10) only that the garbage collection rule is absent and reduction is restricted to a subset of the set of full contexts called *evaluation contexts*. We next describe evaluation contexts. Note that although they rely on a given set of normal forms, for expository purposes, we first describe the evaluation contexts and then characterize its normal forms.

Definition 44. Evaluation context judgments *are expressions of the form $\mathtt{C} \in \mathcal{C}^{h}_{\vartheta}$ where $\mathtt{C}$ is a full context, ϑ is a set of variables and h is a symbol called* discriminator *of the context. This symbol may be one of '$\cdot$', 'λ' or any constant $\mathbf{c}, \mathbf{d}, \ldots$ and will prove convenient to discriminate the head symbol in the context; evaluation context formation rules will place requirements on them. An* **evaluation context** *is a context $\mathtt{C}$ such that the evaluation context judgement $\mathtt{C} \in \mathcal{C}^{h}_{\vartheta}$ is derivable using the rules in Fig. 5.*

eBox states that any redex at the root is needed (we may disregard ϑ and '$\cdot$' for now). Rule eApp-L allows reduction to take place to the left of an application. We must make sure that $\mathtt{C}$ is not an abstraction. This is achieved by requiring that $h \neq \lambda$ (*cf.* eLam and how all rules persist h). Rule eAppRStruct allows reduction to take place to the right of an application when it is an argument of a term t that is a structure normal form or an error normal form. The '$\cdot$' in $t\,\mathtt{C} \in \mathcal{C}^{\cdot}_{\vartheta}$ reflects that t is not headed by a constant and that $t\,\mathtt{C}$ is not an abstraction. Rule eAppRCons is similar only that the discriminator is set to the head variable of t via $\mathsf{hc}(t)$ and will be consulted when deciding if reduction can take place in the condition of a case (*cf.* eCase1). This function is defined as: $\mathsf{hc}(\mathbf{c}) := \mathbf{c}$, $\mathsf{hc}(t\,s) := \mathsf{hc}(t)$ and $\mathsf{hc}(t[x\backslash s]) := \mathsf{hc}(t)$. Note that $\mathsf{hc}(\mathtt{A}[\mathbf{c}]\mathtt{L}) = \mathbf{c}$.

The role of frozen variables is best explained in the setting of eSubsLNonStruct and eSubsLStruct. In a term t such as $x[x\backslash y\,s]$, clearly $y\,s$ is not to be substituted for x since it is not an answer. Thus, computation has to proceed in s. However, if t is placed under an explicit substitution, then whether we should reduce s *depends* on its context. For example, we do want to reduce it in $x[x\backslash y\,s][y\backslash z]$ but not in $x[x\backslash y\,s][y\backslash \lambda z.\mathbf{c}]$ since $\lambda z.\mathbf{c}$ does not use s. These two examples motivate eSubsLStruct (z is a structure normal form) and eSubsLNonStruct ($\lambda z.\mathbf{c}$ is not a structure normal form nor an error term). Also note that in order for the focus of computation to be placed to the right of y in $y\,s$, we must know that y will never be substituted for, or else, that it is *frozen*. Rule eSubs-R allows computation to take place in the body of an explicit substitution.

There is no rule for $\mathtt{fix}(x.t)$ since reduction must take place at the root in a term such as that. Regarding case expressions, in order for reduction to take place in the condition we must ensure that reduction at the root is not possible (*cf.* eCase1). This is achieved by requiring that the discriminator either is not a constant listed in the branches ($h \notin \{\mathbf{c}_i\}_{i\in I}$) or that, if it is, then the number of expected arguments by the branch are not met ($|\mathtt{C}[y]| \neq |\bar{x_j}|$). The notation $|\mathtt{C}[y]|$ counts the number of arguments in the spine of the term $\mathtt{C}[y]$. It is defined as follows:

$$\begin{array}{rcl} |x| & := & 0 \\ |\mathbf{c}| & := & 0 \\ |\lambda x.t| & := & 0 \\ |t\,s| & := & 1+|t| \end{array} \qquad \begin{array}{rcl} |\mathtt{fix}(x.t)| & := & 0 \\ |t[x\backslash s]| & := & |t| \\ |\mathtt{case}\ t\ \mathtt{of}\ \bar{b}| & := & 0 \end{array}$$

We know that in fact $\mathtt{C}[y]$ is a constant answer:

Lemma 45 (Answer contexts are answers)**.** *Suppose* $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$*.*

- *If* $h = \mathbf{c}$*, then, for any term* t*, there exist* $\mathtt{A}$ *and* $\mathtt{L}$ *s.t.* $\mathtt{C}[t] = \mathtt{A}[\mathbf{c}]\mathtt{L}$*.*
- *If* $h = \lambda$*, then, for any term* t*, there exists a variable* x*, term* s *and substitution context* $\mathtt{L}$ *s.t.* $\mathtt{C}[t] = (\lambda x.s)\mathtt{L}$*. Moreover,* $\mathtt{C}$ *is either of the form* $(\lambda x.\mathtt{C}')\mathtt{L}$ *or* $(\lambda x.t)\mathtt{L}_1[y\backslash \mathtt{C}']\mathtt{L}_2$*.*

Proof. By induction on the derivation of $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$. Note that since $h \in \{\mathbf{c}, \lambda\}$, this derivation cannot end in any of the rules eBox, eAppRStruct, eCase1, or eCase2. The remaining cases are addressed below.

- eAppL. The derivation is as follows:

$$\frac{\mathtt{C} \in \mathcal{C}^h_{\vartheta} \quad h \neq \lambda}{\mathtt{C}\,s \in \mathcal{C}^h_{\vartheta}}\ \textsc{eAppL}$$

  Then the first item holds and by the *i.h.* there exist $\mathtt{A}'$ and $\mathtt{L}'$ s.t. $\mathtt{C}[t] = \mathtt{A}'[\mathbf{c}]\mathtt{L}'$. We set $\mathtt{A} := \mathtt{A}'\mathtt{L}'\,s$ and $\mathtt{L} := \epsilon$.

- eAppRCons. The derivation is as follows:

$$\frac{t \in \mathcal{K}_\vartheta \quad \mathtt{C} \in \mathcal{C}^h_\vartheta}{t\,\mathtt{C} \in \mathcal{C}^{\mathsf{hc}(t)}_\vartheta}\ \text{eAppRCons}$$

  The first item holds. The result follows from Lem. 60.

- eSubsLNonStruct. The derivation is as follows:

$$\frac{\mathtt{C} \in \mathcal{C}^h_\vartheta \quad s \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta \quad x \notin \vartheta}{\mathtt{C}[x\backslash s] \in \mathcal{C}^h_\vartheta}\ \text{eSubsLNonStruct}$$

  - If $h = \mathbf{c}$, by the *i.h.*, for any t, there exist $\mathtt{A}'$ and $\mathtt{L}'$ s.t. $\mathtt{C}[t] = \mathtt{A}'[\mathbf{c}]\mathtt{L}'$. We set $\mathtt{A} := \mathtt{A}'$ and $\mathtt{L} := \mathtt{L}'[x\backslash s]$.
  - If $h = \lambda$, by the *i.h.* there exists a variable y, term u' and substitution context $\mathtt{L}'$ s.t. $\mathtt{C}[t] = (\lambda y.u')\mathtt{L}'$. Moreover, $\mathtt{C}$ is either of the form $(\lambda y.\mathtt{C}')\mathtt{L}'$ or $(\lambda y.u')\mathtt{L}_1[y\backslash \mathtt{C}']\mathtt{L}_2$. We set $u = u'$ and $\mathtt{L} = \mathtt{L}'[x\backslash s]$ and conclude.

- eSubsLStruct. The derivation is as follows:

$$\frac{\mathtt{C} \in \mathcal{C}^h_{\vartheta\cup\{x\}} \quad t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta}{\mathtt{C}[x\backslash s] \in \mathcal{C}^h_\vartheta}\ \text{eSubsLStruct}$$

  - If $h = \mathbf{c}$, then by the *i.h.*, for any t, there exist $\mathtt{A}'$ and $\mathtt{L}'$ s.t. $\mathtt{C}[t] = \mathtt{A}'[\mathbf{c}]\mathtt{L}'$. We set $\mathtt{A} := \mathtt{A}'$ and $\mathtt{L} := \mathtt{L}'[x\backslash s]$.
  - If $h = \lambda$, by the *i.h.* there exists a variable y, term u' and substitution context $\mathtt{L}'$ s.t. $\mathtt{C}[t] = (\lambda y.u')\mathtt{L}'$. Moreover, $\mathtt{C}$ is either of the form $(\lambda y.\mathtt{C}')\mathtt{L}'$ or $(\lambda y.u')\mathtt{L}_1[y\backslash \mathtt{C}']\mathtt{L}_2$. We set $u = u'$ and $\mathtt{L} = \mathtt{L}'[x\backslash s]$ and conclude.

- eSubsR. The derivation is as follows:

$$\frac{\mathtt{C}_1 \in \mathcal{C}^h_\vartheta \quad \mathtt{C}_2 \in \mathcal{C}^\cdot_\vartheta}{\mathtt{C}_1[\![x]\!][x\backslash \mathtt{C}_2] \in \mathcal{C}^h_\vartheta}\ \text{eSubsR}$$

  - If $h = \mathbf{c}$, then by the *i.h.* on the derivation of $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$ with $t = x$, we deduce that there exist $\mathtt{A}'$ and $\mathtt{L}'$ s.t. $\mathtt{C}_1[\![x]\!] = \mathtt{A}'[\mathbf{c}]\mathtt{L}'$. Given any t, we set $\mathtt{A} := \mathtt{A}'$ and $\mathtt{L} := \mathtt{L}'[x\backslash \mathtt{C}_2[t]]$.
  - If $h = \lambda$, by the *i.h.*, we pick $t = x$ and hence there exists a variable y, term u' and substitution context $\mathtt{L}'$ s.t. $\mathtt{C}[\![x]\!] = (\lambda y.u')\mathtt{L}'$. Moreover, $\mathtt{C}$ is either of the form $(\lambda y.\mathtt{C}')\mathtt{L}'$ or $(\lambda y.u')\mathtt{L}_1[y\backslash \mathtt{C}']\mathtt{L}_2$. Given any t, we set $u = u'$ and $\mathtt{L} = \mathtt{L}'[x\backslash \mathtt{C}_2[t]]$ and conclude.

- eLam. The derivation is as follows:

$$\frac{\mathtt{C} \in \mathcal{C}^h_{\vartheta\cup\{y\}}}{\lambda y.\mathtt{C} \in \mathcal{C}^\lambda_\vartheta}\ \text{eLam}$$

  The second item holds. We conclude immediately (with $\mathtt{L} := \epsilon$).

□

For reduction to proceed in a branch j (*cf.* eCase2), the condition must be in normal form, each branch i with $i \in 1..j$ must be in normal form and the condition must not enable any branch $(t \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I})$. Note that the bound variables in branch j, are added to the set of frozen variables. We now define the strategy itself.

Definition 46. *The $\rightarrowtail^{\vartheta}_{\mathtt{sh}}$ strategy is defined by the following rules.*

$$\begin{array}{rcll}
\mathtt{C}[(\lambda x.t)\mathtt{L}\, s] & \rightarrowtail^{\vartheta}_{\mathtt{sh}} & \mathtt{C}[t[x\backslash s]\mathtt{L}] & (\mathtt{dB}) \\
 & & \textit{if } \mathtt{C} \in \mathcal{C}^{h}_{\vartheta} & \\
\mathtt{C}_1[\mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}]] & \rightarrowtail^{\vartheta}_{\mathtt{sh}} & \mathtt{C}_1[\mathtt{C}_2[v][x\backslash v]\mathtt{L}] & (\mathtt{lsv}) \\
 & & \textit{if } \mathtt{C}_1[\mathtt{C}_2[\Box][x\backslash v\mathtt{L}]] \in \mathcal{C}^{h}_{\vartheta} & \\
\mathtt{C}[\mathtt{fix}(x.t)] & \rightarrowtail^{\vartheta}_{\mathtt{sh}} & \mathtt{C}[t[x\backslash \mathtt{fix}(x.t)]] & (\boldsymbol{\mathit{fix}}) \\
\mathtt{C}[\mathtt{case}\ \mathtt{A}[\mathbf{c}_j]\mathtt{L}\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}] & \rightarrowtail^{\vartheta}_{\mathtt{sh}} & \mathtt{C}[s_j[\bar{x}_j\backslash \mathtt{A}]\mathtt{L}] & (\boldsymbol{\mathit{case}}) \\
 & & \textit{if } \mathtt{C} \in \mathcal{C}^{h}_{\vartheta} \textit{ and } j \in I \textit{ and } |\mathtt{A}[\Box]| = |\bar{x}_j| &
\end{array}$$

The discriminator h in the conditions of all rules is existentially quantified. The condition $\mathtt{C}_1[\mathtt{C}_2[\Box][x\backslash v\mathtt{L}]] \in \mathcal{C}^{h}_{\vartheta}$ in the definition of the $\mathtt{lsv}$-redex carries over from [BBBK17]. It avoids $\mathtt{lsv}$-reducing $(\lambda x.y)[y\backslash \mathtt{id}]$ in $(\lambda x.y)[y\backslash \mathtt{id}]t$ so that the outermost $\mathtt{dB}$-reduction step takes precedence instead. The condition also avoids to $\mathtt{lsv}$-reduce $x[x\backslash(\lambda y.yz)[z\backslash \mathtt{id}]]$ on the variable z, so that the $\mathtt{lsv}$-reduction step on the variable x takes precedence over it. The following result states that the strategy is deterministic:

Lemma 47 (Determinism)**.** *If $\mathtt{C}_1[r_1] = \mathtt{C}_2[r_2]$, where $\mathtt{C}_1, \mathtt{C}_2 \in \mathcal{C}^{h}_{\vartheta}$ and r_1, r_2 are redexes, then $\mathtt{C}_1 = \mathtt{C}_2$ and $r_1 = r_2$.*

Determinism is a consequence of a slightly more general result, namely Lem. 58. The proof of the more general result requires introducing some preliminary notions and proofs.

Definition 48 (Variables frozen by an evaluation context)**.** *Given an evaluation context $\mathtt{C} \in \mathcal{C}^{h}_{\vartheta}$, we write $\mathsf{fz}^{\vartheta}(\mathtt{C})$ for the set ϑ extended with all variables bound by abstractions, or bound to weak structures or weak error terms, in the path from the root to the hole of $\mathtt{C}$. This is defined by induction in the judgement $\mathtt{C} \in \mathcal{C}^{h}_{\vartheta}$* (cf. *Fig. 5):*

$$\begin{array}{rcll}
\mathsf{fz}^{\vartheta}(\Box) & := & \vartheta & (\text{eBox}) \\
\mathsf{fz}^{\vartheta}(\mathtt{C}t) & := & \mathsf{fz}^{\vartheta}(\mathtt{C}) & (\text{eAppL}) \\
\mathsf{fz}^{\vartheta}(t\,\mathtt{C}) & := & \mathsf{fz}^{\vartheta}(\mathtt{C}) & (\text{eAppRStruct}) \\
\mathsf{fz}^{\vartheta}(t\,\mathtt{C}) & := & \mathsf{fz}^{\vartheta}(\mathtt{C}) & (\text{eAppRCons}) \\
\mathsf{fz}^{\vartheta}(\mathtt{C}[x\backslash t]) & := & \mathsf{fz}^{\vartheta}(\mathtt{C}) & (\text{eSubsLNonStruct}) \\
\mathsf{fz}^{\vartheta}(\mathtt{C}[x\backslash t]) & := & \mathsf{fz}^{\vartheta\cup\{x\}}(\mathtt{C}) & (\text{eSubsLStruct}) \\
\mathsf{fz}^{\vartheta}(\mathtt{C}_1[\![x]\!][x\backslash \mathtt{C}_2]) & := & \mathsf{fz}^{\vartheta}(\mathtt{C}_2) & (\text{eSubsR}) \\
\mathsf{fz}^{\vartheta}(\lambda x.\mathtt{C}) & := & \mathsf{fz}^{\vartheta\cup\{x\}}(\mathtt{C}) & (\text{eLam}) \\
\mathsf{fz}^{\vartheta}(\mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n) & := & \mathsf{fz}^{\vartheta\cup\{\bar{x_j}\}}(\mathtt{C}) & (\text{eCase1}) \\
\mathsf{fz}^{\vartheta}(\mathtt{case}\ \mathtt{C}\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}) & := & \mathsf{fz}^{\vartheta}(\mathtt{C}) & (\text{eCase2})
\end{array}$$

Lemma 49. $\mathsf{fz}^{\vartheta}(\mathtt{C}_1[\mathtt{C}_2]) = \mathsf{fz}^{\mathsf{fz}^{\vartheta}(\mathtt{C}_1)}(\mathtt{C}_2)$

Proof. By induction on $\mathtt{C}_1$. ☐

Lemma 50 (Decomposition of evaluation contexts)**.** *If $\mathtt{C}_1[\mathtt{C}_2] \in \mathcal{C}^h_\vartheta$ then $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$ and $\mathtt{C}_2 \in \mathcal{C}^{h'}_{\vartheta'}$, for some h' and where $\vartheta' = \mathsf{fz}^\vartheta(\mathtt{C}_1)$.*

Proof. If $\mathtt{C}_1$ is empty, it is immediate that $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$ and $\mathtt{C}_2 \in \mathcal{C}^{h'}_{\vartheta'}$ with $h' = h$ and $\vartheta' = \vartheta = \mathsf{fz}^\vartheta(\mathtt{C}_1)$, so we may assume that $\mathtt{C}_1 \neq \Box$. We proceed by induction on $\mathtt{C}_1[\mathtt{C}_2] \in \mathcal{C}^h_\vartheta$.

1. **Empty, $\mathtt{C}_1[\mathtt{C}_2] = \Box$.** Then $\mathtt{C}_1$ is empty, so it is immediate.
2. **Left of an application, $\mathtt{C}_1[\mathtt{C}_2] = \mathtt{C}'_1[\mathtt{C}_2]\,t$ with $\mathtt{C}'_1[\mathtt{C}_2] \in \mathcal{C}^h_\vartheta$ and $h \neq \lambda$.** By *i.h.* $\mathtt{C}'_1 \in \mathcal{C}^h_\vartheta$, so $\mathtt{C}_1 = \mathtt{C}'_1\,t \in \mathcal{C}^h_\vartheta$. Also by *i.h.*, $\mathtt{C}_2 \in \mathcal{C}^{h'}_{\vartheta'}$ with $\vartheta' = \mathsf{fz}^\vartheta(\mathtt{C}'_1) = \mathsf{fz}^\vartheta(\mathtt{C}_1)$.
3. **Non-structural substitution, $\mathtt{C}_1[\mathtt{C}_2] = \mathtt{C}'_1[\mathtt{C}_2][x\backslash t]$ with $\mathtt{C}'_1 \in \mathcal{C}^h_\vartheta$, $t \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$, $x \notin \vartheta$.** By *i.h.* $\mathtt{C}'_1 \in \mathcal{C}^h_\vartheta$, so $\mathtt{C}_1 = \mathtt{C}'_1[x\backslash t] \in \mathcal{C}^h_\vartheta$. Also by *i.h.*, $\mathtt{C}_2 \in \mathcal{C}^{h'}_{\vartheta'}$ with $\vartheta' = \mathsf{fz}^\vartheta(\mathtt{C}'_1) = \mathsf{fz}^\vartheta(\mathtt{C}_1)$.
4. **Structural substitution, $\mathtt{C}_1[\mathtt{C}_2] = \mathtt{C}'_1[\mathtt{C}_2][x\backslash t]$ with $\mathtt{C}'_1[\mathtt{C}_2] \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ and $t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** By *i.h.* $\mathtt{C}'_1 \in \mathcal{C}^h_{\vartheta\cup\{x\}}$, so $\mathtt{C}_1 = \mathtt{C}'_1[x\backslash t] \in \mathcal{C}^h_\vartheta$. Also by *i.h.*, $\mathtt{C}_2 \in \mathcal{C}^{h'}_{\vartheta'}$ for some h' with $\vartheta' = \mathsf{fz}^{\vartheta\cup\{x\}}(\mathtt{C}'_1) = \mathsf{fz}^\vartheta(\mathtt{C}_1)$.
5. **Inside a substitution, $\mathtt{C}_1[\mathtt{C}_2] = \mathtt{C}[\![x]\!][x\backslash \mathtt{C}'_1[\mathtt{C}_2]]$ with $\mathtt{C}'_1[\mathtt{C}_2] \in \mathcal{C}^h_\vartheta$.** By *i.h.* $\mathtt{C}'_1 \in \mathcal{C}^h_\vartheta$ so $\mathtt{C}_1 = \mathtt{C}[\![x]\!][x\backslash \mathtt{C}'_1] \in \mathcal{C}^h_\vartheta$. Also by *i.h.*, $\mathtt{C}_2 \in \mathcal{C}^{h'}_{\vartheta'}$ for some h' with $\vartheta' = \mathsf{fz}^\vartheta(\mathtt{C}'_1) = \mathsf{fz}^\vartheta(\mathtt{C}_1)$.
6. **Right of a structure, $\mathtt{C}_1[\mathtt{C}_2] = t\,\mathtt{C}'_1[\mathtt{C}_2]$ with $\mathtt{C}'_1[\mathtt{C}_2] \in \mathcal{C}^h_\vartheta$ and $t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ and $h = \cdot$.** By *i.h.* $\mathtt{C}'_1 \in \mathcal{C}^h_\vartheta$ so $\mathtt{C}_1 = t\,\mathtt{C}'_1 \in \mathcal{C}^{\cdot}_\vartheta$. Also by *i.h.*, $\mathtt{C}_2 \in \mathcal{C}^{h'}_{\vartheta'}$ for some h' with $\vartheta' = \mathsf{fz}^\vartheta(\mathtt{C}'_1) = \mathsf{fz}^\vartheta(\mathtt{C}_1)$.
7. **Right of a constructor answer, $\mathtt{C}_1[\mathtt{C}_2] = t\,\mathtt{C}'_1[\mathtt{C}_2]$ with $\mathtt{C}'_1[\mathtt{C}_2] \in \mathcal{C}^h_\vartheta$ and $t \in \mathcal{K}_\vartheta$.** By *i.h.* $\mathtt{C}'_1 \in \mathcal{C}^h_\vartheta$ so $\mathtt{C}_1 = t\,\mathtt{C}'_1 \in \mathcal{C}^h_\vartheta$. Also by *i.h.*, $\mathtt{C}_2 \in \mathcal{C}^{h'}_{\vartheta'}$ for some h' and $\vartheta' = \mathsf{fz}^\vartheta(\mathtt{C}'_1) = \mathsf{fz}^\vartheta(\mathtt{C}_1)$.
8. **Under an abstraction, $\mathtt{C}_1[\mathtt{C}_2] = \lambda x.\mathtt{C}'_1[\mathtt{C}_2]$ with $\mathtt{C}'_1[\mathtt{C}_2] \in \mathcal{C}^{h''}_{\vartheta\cup\{x\}}$, for some h'', and $h = \lambda$.** By *i.h.* $\mathtt{C}'_1 \in \mathcal{C}^{h''}_{\vartheta\cup\{x\}}$ so $\mathtt{C}_1 = \lambda x.\mathtt{C}'_1 \in \mathcal{C}^\lambda_\vartheta$. Also by *i.h.*, $\mathtt{C}_2 \in \mathcal{C}^{h'}_{\vartheta'}$ for some h' with $\vartheta' = \mathsf{fz}^{\vartheta\cup\{x\}}(\mathtt{C}'_1) = \mathsf{fz}^\vartheta(\mathtt{C}_1)$.
9. **In the branch of a case, $\mathtt{C}_1[\mathtt{C}_2] = \mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}'_1[\mathtt{C}_2], \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n$ and $t \in \mathcal{N}_\vartheta$ and $t \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ and $t_k \in \mathcal{N}_{\vartheta\cup\bar{x_k}}$ for all $k < j$ and $\mathtt{C}'_1[\mathtt{C}_2] \in \mathcal{C}^{h''}_{\vartheta\cup\bar{x_i}}$ and $h = \cdot$.** By *i.h.* $\mathtt{C}'_1 \in \mathcal{C}^{h''}_{\vartheta\cup\{x\}}$ so $\mathtt{C}_1 = \mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}'_1, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n \in \mathcal{C}^{\cdot}_\vartheta$. Also by *i.h.*, $\mathtt{C}_2 \in \mathcal{C}^{h'}_{\vartheta'}$ for some h' with $\vartheta' = \mathsf{fz}^{\vartheta\cup\{\bar{x_j}\}}(\mathtt{C}'_1) = \mathsf{fz}^\vartheta(\mathtt{C}_1)$.
10. **In the condition of a case, $\mathtt{C}_1[\mathtt{C}_2] = \mathtt{case}\ \mathtt{C}'_1[\mathtt{C}_2]\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{C}^{\cdot}_\vartheta$ and $\mathtt{C}_1[\mathtt{C}_2] \in \mathcal{C}^{h''}_\vartheta$ and $h'' \notin \{\mathbf{c}_i\}_{i\in I}$ or $h'' = \mathbf{c}_j \in \{\mathbf{c}_i\}_{i\in I}$ and $|\mathtt{A}(\mathtt{C}, y)| \neq |\bar{x_j}|$ and $h = \cdot$.** By *i.h.* $\mathtt{C}'_1 \in \mathcal{C}^{h''}_\vartheta$ so $\mathtt{C}_1 = \mathtt{case}\ \mathtt{C}'_1\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{C}^{\cdot}_\vartheta$. Also by *i.h.*, $\mathtt{C}_2 \in \mathcal{C}^{h'}_{\vartheta'}$ for some h' with $\vartheta' = \mathsf{fz}^\vartheta(\mathtt{C}'_1) = \mathsf{fz}^\vartheta(\mathtt{C}_1)$.

$\Box$

Definition 51 (Reduction place)**.** *In a term $\mathtt{C}[t]$, with $\mathtt{C} \in \mathcal{C}^h_\vartheta$ for some h, the subterm t is said to be a* **$\mathtt{C}$-reduction place** *if any of the following hold:*

1. *t is the redex pattern of a beta-step,* i.e. $t = (\lambda x.s)\mathtt{L}\,u$;

2. *t is the variable contracted by an ls-step,* i.e. $t = x$ *and* $\mathtt{C} = \mathtt{C}_1[\mathtt{C}_2[x \backslash v\mathtt{L}]]$, *where* $\mathtt{C}_2 \in \mathcal{C}^{h'}_{\vartheta'}$ *for some* h', *and* $\vartheta' = \mathsf{fz}^{\vartheta}(\mathtt{C}_1)$*;*

3. *t is a free variable (not bound by* $\mathtt{C}$*) such that* $x \notin \mathsf{fz}^{\vartheta}(\mathtt{C})$*;*

4. *t is the redex pattern of a fix-step,* i.e. $t = \mathtt{fix}(x.s)$*;*

5. *t is the redex pattern of a case-step,* i.e. $t = \mathtt{case}\ \mathtt{A}[\mathbf{c}_j]\mathtt{L}\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i \in I}$ *and* $|\mathtt{A}[\mathbf{c}_j]\mathtt{L}| = |\bar{x}_j|$ *and* $j \in I$.

Lemma 52 (Reduction places are stable by trimming a context down)**.** *Let* $\mathtt{C}_1[\mathtt{C}_2] \in \mathcal{C}^h_{\vartheta}$*, and let t be a* $\mathtt{C}_1[\mathtt{C}_2]$*-reduction place. Then t is a* $\mathtt{C}_2$*-reduction place.*

Proof. Let us consider the five cases in Def. 51 for the fact that t is a $\mathtt{C}_1[\mathtt{C}_2]$-reduction place:

1. **If t is the redex pattern of a beta-step.** Then t is trivially a $\mathtt{C}_2$-reduction place, as being the redex pattern of a beta-step does not depend on the context.

2. **If t is the variable contracted by an ls-step.** That is, $t = x$ and x is bound to an answer $v\mathtt{L}$. There are two cases, depending on whether x is bound by the external context $\mathtt{C}_1$ or by the internal context $\mathtt{C}_2$:

   (a) *If x is bound by* $\mathtt{C}_1$.
   Then x is not bound by $\mathtt{C}_2$. To show that $t = x$ is indeed a $\mathtt{C}_2$-reduction place, it suffices to show that $x \notin \mathsf{fz}^{\vartheta'}(\mathtt{C}_2)$. By Lem. 49 we know that $\mathsf{fz}^{\vartheta'}(\mathtt{C}_2) = \mathsf{fz}^{\vartheta}(\mathtt{C}_1[\mathtt{C}_2])$. Since x is bound by $\mathtt{C}_1$, let us write $\mathtt{C}_1 = \mathtt{C}_{11}[\mathtt{C}_{12}[x \backslash v\mathtt{L}]]$. We know that $x \notin \vartheta$ by Barendregt's convention. By applying Lem. 49 again we obtain that $\mathsf{fz}^{\vartheta}(\mathtt{C}_1[\mathtt{C}_2]) = \mathsf{fz}^{\vartheta'''}(\mathtt{C}_{12}[\mathtt{C}_2][x \backslash v\mathtt{L}])$, where $\vartheta''' = \mathsf{fz}^{\vartheta}(\mathtt{C}_{11})$. Note that x is not bound by $\mathtt{C}_{11}$, so $x \notin \vartheta'''$.

   Now note that $v\mathtt{L}$ is an answer but not a structure, so $\vartheta''' = \vartheta''$ and $\mathsf{fz}^{\vartheta'''}(\mathtt{C}_{12}[\mathtt{C}_2][x \backslash v\mathtt{L}]) = \mathsf{fz}^{\vartheta''}(\mathtt{C}_{12}[\mathtt{C}_2])$. Note also that since $x \notin \vartheta'''$ and x is not bound by $\mathtt{C}_{12}[\mathtt{C}_2]$ we know that $x \notin \mathsf{fz}^{\vartheta''}(\mathtt{C}_{12}[\mathtt{C}_2])$. Finally, we may apply Lem. 49 once more to conclude that $x \notin \mathsf{fz}^{\vartheta''}(\mathtt{C}_{12}[\mathtt{C}_2]) = \mathsf{fz}^{\vartheta'}(\mathtt{C}_2)$, by which we conclude that x is a $\mathtt{C}_2$-reduction place, as required.

   (b) *If x is bound by* $\mathtt{C}_2$.
   Then $t = x$ is trivially a $\mathtt{C}_2$-reduction place, as it is the variable contracted by an ls-step.

3. **If t is a free variable x such that $x \notin \mathsf{fz}^{\vartheta}(\mathtt{C}_1[\mathtt{C}_2])$.** As x is not bound by $\mathtt{C}_1[\mathtt{C}_2]$, we have that x is also not bound by $\mathtt{C}_2$. Moreover, by Lem. 49 we have that $\mathsf{fz}^{\vartheta}(\mathtt{C}_1[\mathtt{C}_2]) = \mathsf{fz}^{\mathsf{fz}^{\vartheta}(\mathtt{C}_1)}(\mathtt{C}_2)$. Since the composition $\mathtt{C}_1[\mathtt{C}_2]$ is a context in $\mathcal{C}^h_{\vartheta}$, by the decomposition of evaluation contexts (Lem. 50) we know that $\mathsf{fz}^{\vartheta}(\mathtt{C}_1) = \vartheta'$, so we conclude that $x \notin \mathsf{fz}^{\vartheta}(\mathtt{C}_1[\mathtt{C}_2]) = \mathsf{fz}^{\vartheta'}(\mathtt{C}_2)$, so t is a $\mathtt{C}_2$-reduction place, as required.

4. **If t is the redex pattern of a fix-step, *i.e.* $t = \mathtt{fix}(x.s)$.** Then t is trivially a $\mathtt{C}_2$-reduction place, as being the redex pattern of a fix-step does not depend on the context.

5. **If t is the redex pattern of a case-step, *i.e.* $t = \mathtt{case}\ \mathtt{A}[\mathbf{c}_j]\mathtt{L}\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i \in I}$ and $|\mathtt{A}[\mathbf{c}_j]\mathtt{L}| = |\bar{x}_j|$ and $j \in I$.** Then t is trivially a $\mathtt{C}_2$-reduction place, as being the redex pattern of a case-step does not depend on the context.

$\square$

Lemma 53 (Non-abstraction answer evaluation contexts do not go below abstraction answers)**.** *Let $t = v\mathtt{L}$ be an answer. Suppose that $t = \mathtt{C}[t']$ for some context $\mathtt{C} \in \mathcal{C}_{\vartheta}^{\cdot}$, some set of variables ϑ, and some term t'. Then $\mathtt{C}$ is a substitution context,* i.e. *$\mathtt{L}$ can be split as $\mathtt{L} = \mathtt{L}_1\mathtt{L}_2$ such that $\mathtt{C} = \mathtt{L}_2$.*

Proof. By induction on the length of the substitution context $\mathtt{L}$.

1. **Empty, $\mathtt{L} = \square$.** Immediate, by noting that no formation rule for $\mathtt{C}$ allows going below an abstraction except ELam (which concludes with $h = \lambda$), so $\mathtt{C}$ must be empty.

2. **Non-empty, $\mathtt{L} = \mathtt{L}'[y\backslash u]$.** We consider three cases, depending on the formation rule applied to build the context $\mathtt{C}$:

   (a) **Non-structural substitution, *i.e.* $u \notin \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$, $y \notin \vartheta$ and $\mathtt{C} = \mathtt{C}_1[y\backslash u]$ with $\mathtt{C}_1 \in \mathcal{C}_{\vartheta}^{h}$.** Note that $v\mathtt{L}' = \mathtt{C}_1[t']$, so by *i.h.* $\mathtt{C}_1$ must be a substitution context. Then $\mathtt{C} = \mathtt{C}_1[y\backslash u]$ is also a substitution context.

   (b) **Structural substitution, *i.e.* $u \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$ and $\mathtt{C} = \mathtt{C}_1[y\backslash u]$ with $\mathtt{C}_1 \in \mathcal{C}_{\vartheta \cup \{y\}}^{h}$.** Similar to the previous case.

   (c) **Inside the substitution, *i.e.* $\mathtt{C} = \mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}_2]$ and $\mathtt{C}_1 \in \mathcal{C}_{\vartheta}^{\cdot}$ and $\mathtt{C}_2 \in \mathcal{C}_{\vartheta}^{\cdot}$.** Note that $v\mathtt{L}' = \mathtt{C}_1[\![y]\!]$, so by *i.h.* $\mathtt{C}_1$ must be a substitution context $\mathtt{L}''$. This implies $v\mathtt{L}' = \mathtt{C}_1[\![y]\!] = y\mathtt{L}''$, which is a contradiction.

$\square$

Corollary 54 (Evaluation contexts do not go below beta-steps)**.** *Let $t = (\lambda x.s)\mathtt{L}\, u$ be the redex pattern of a beta-step. Suppose that $t = \mathtt{C}[t']$ for some non-empty context $\mathtt{C} \in \mathcal{C}_{\vartheta}^{h}$, some set of variables ϑ, and some term t'. Then $\mathtt{L}$ can be split as $\mathtt{L} = \mathtt{L}_1\mathtt{L}_2$ such that $\mathtt{C} = \mathtt{L}_2\, u$.*

Proof. We consider the three possible formation rules for $\mathtt{C}$ as a context in $\mathcal{C}_{\vartheta}^{h}$:

1. **Left of an application (eAppL), *i.e.* $\mathtt{C} = \mathtt{C}_1\, r$.** Then $r = u$. By the previous Lem. 53 we have that $\mathtt{C}_1$ is a substitution context, and we conclude.

2. **Right of a weak structure or error term (eAppRStruct), *i.e.* $\mathtt{C} = r\,\mathtt{C}_1$ and $r \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$.** Impossible, as $(\lambda x.s)\mathtt{L} \notin \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$.

3. **Right of a constructor answer (eAppRCons), *i.e.* $\mathtt{C} = r\,\mathtt{C}_1$ and $r \in \mathcal{K}_{\vartheta}$.** Impossible, as $(\lambda x.s)\mathtt{L} \notin \mathcal{K}_{\vartheta}$.

$\square$

Lemma 55. *If $\mathtt{A}[\mathbf{c}]\mathtt{L} = \mathtt{C}[t] \in \mathcal{C}_{\vartheta}^{h}$ and t is a variable, beta-redex, fix-redex or case-redex, then $h = \mathbf{c}$.*

Proof. By induction on $\mathtt{C}[t] \in \mathcal{C}_{\vartheta}^{h}$.

- **eBox.** Not possible since $\mathtt{A}[\mathbf{c}]\mathtt{L}$ is not a variable, beta-redex, fix or case expression.

- **eCase1, eCase2 and eLam.** Not possible since $\mathtt{A}[\mathbf{c}]\mathtt{L}$ is not a case expression nor a lambda expression.

- **eAppL.** Follows from the *i.h.*

$$\frac{\mathtt{C} \in \mathcal{C}^h_\vartheta \quad h \neq \lambda}{\mathtt{C}\, t \in \mathcal{C}^h_\vartheta}\ \text{eAppL}$$

- **eAppRStruct.**. Not possible since $t \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.

$$\frac{t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta \quad \mathtt{C} \in \mathcal{C}^h_\vartheta}{t\, \mathtt{C} \in \mathcal{C}^{\cdot}_\vartheta}\ \text{eAppRStruct}$$

- **eAppRCons.** Then $\mathtt{L} = \epsilon$ and $\mathtt{A} = \mathtt{A}_1\, u$ and $s = \mathtt{A}_1[\mathbf{c}]$. We conclude from the fact that $\mathsf{hc}(\mathtt{A}_1[\mathbf{c}]) = \mathbf{c}$.

$$\frac{s \in \mathcal{K}_\vartheta \quad \mathtt{C} \in \mathcal{C}^h_\vartheta}{s\, \mathtt{C} \in \mathcal{C}^{\mathsf{hc}(t)}_\vartheta}\ \text{eAppRCons}$$

- **eSubsLNonStruct.** From the *i.h.*

$$\frac{\mathtt{C} \in \mathcal{C}^h_\vartheta \quad t \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta \quad x \notin \vartheta}{\mathtt{C}[x\backslash t] \in \mathcal{C}^h_\vartheta}\ \text{eSubsLNonStruct}$$

- **eSubsLStruct.** From the *i.h.*

$$\frac{\mathtt{C} \in \mathcal{C}^h_{\vartheta \cup \{x\}} \quad t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta}{\mathtt{C}[x\backslash t] \in \mathcal{C}^h_\vartheta}\ \text{eSubsLStruct}$$

- **eSubsR.** Then $\mathtt{A}[\mathbf{c}]\mathtt{L} = \mathtt{C}_1[\![x]\!][x\backslash \mathtt{C}_2[t]]$ implies $\mathtt{L} = \mathtt{L}'[x\backslash \mathtt{C}_2[t]]$ and $\mathtt{C}_1[\![x]\!] = \mathtt{A}_1[\mathbf{c}]\mathtt{L}'$.

$$\frac{\mathtt{C}_1 \in \mathcal{C}^h_\vartheta \quad \mathtt{C}_2 \in \mathcal{C}^{\cdot}_\vartheta}{\mathtt{C}_1[\![x]\!][x\backslash \mathtt{C}_2] \in \mathcal{C}^h_\vartheta}\ \text{eSubsR}$$

  We thus conclude from the *i.h.* on $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$ and taking $t = x$.

□

Lemma 56 (Evaluation contexts do not go below matching cases)**.** *Let* $t = \mathtt{case}\ \mathtt{A}[\mathbf{c}_j]\mathtt{L}\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ *be the redex pattern of a case-step. Suppose that* $t = \mathtt{C}[t']$ *for some non-empty context* $\mathtt{C} \in \mathcal{C}^h_\vartheta$*, some* h*, set of variables* ϑ*, and some term* t' *that is a* $\mathtt{C}$*-reduction place. Then either* $j \notin I$ *or* $|\mathtt{A}[\mathbf{c}_j]\mathtt{L}| \neq |\bar{x}_j|$.

Proof. There are only two cases:

1. **eCase1 (the branch of a case).** Then the condition $\mathtt{A}[\mathbf{c}_j]\mathtt{L} \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ of eCase1 of implies either $j \notin I$ or $|\mathtt{A}[\mathbf{c}_j]\mathtt{L}| \neq |\bar{x}_j|$.
2. **eCase2 (the condition of a case).** Then the following conditions hold: $\mathtt{A}[\mathbf{c}_j]\mathtt{L} = \mathtt{C}_1[t'] \in \mathcal{C}^h_\vartheta$ and $h \notin \{\mathbf{c}_i\}_{i\in I}$ or $h = \mathbf{c}_j \in \{\mathbf{c}_i\}_{i\in I}$ and $|\mathtt{A}(\mathtt{C}, y)| \neq |\bar{x_j}|$. The result follows from Lem. 55, which establishes that h must be a constant.

$\square$

Lemma 57 (Strong normal forms have no reduction places under an evaluation context)**.** *Let $r \in \mathcal{N}_\vartheta$ be a strong normal form. Then r cannot be written as $\mathtt{C}[t]$ such that $\mathtt{C} \in \mathcal{C}^h_\vartheta$ and t is a $\mathtt{C}$-reduction place.*

Proof. Suppose that $r = \mathtt{C}[t]$, where t is a $\mathtt{C}$-reduction place. Let us check that this is impossible by induction on $\mathtt{C} \in \mathcal{C}^h_\vartheta$:

1. **Empty, *i.e.* $\mathtt{C} = \square$.** Then r must be a $\square$-reduction place, for $\square$ as a context in $\mathcal{C}^h_\vartheta$. Let us consider the five cases of the definition of $\square$-reduction place:

   (a) **If r is the redex pattern of a beta-step.** The only way to derive $r \in \mathcal{N}_\vartheta$ would be by having $r = s\, r_1$ with s an answer. But strong structures are not answers, so this case is impossible.

   (b) **If r is the variable x contracted by an ls-step.** Impossible, as there are no substitutions to bind x.

   (c) **If r is a free variable x such that $x \notin \mathtt{fz}^\vartheta(\square)$.** Impossible, as this means that $x \notin \vartheta$, but a variable x is a strong normal form in $\mathcal{N}_\vartheta$ if and only if $x \in \vartheta$.

   (d) **r is the redex pattern of a fix-step, *i.e.* $t = \mathtt{fix}(x.s)$.** It is not possible to derive $r \in \mathcal{N}_\vartheta$.

   (e) **r is the redex pattern of a case-step, *i.e.* $t = \mathtt{case}\ \mathtt{A}[\mathbf{c}_j]\mathtt{L}\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ and $|\mathtt{A}[\mathbf{c}_j]\mathtt{L}| = |\bar{x}_j|$ and $j \in I$.** $r \in \mathcal{N}_\vartheta$ can only be derived using eNfCase or eNfStrt. The former is not possible since $\mathtt{A}[\mathbf{c}_j]\mathtt{L} \notin \mathcal{E}_\vartheta$ and the latter is not possible since the condition $\mathtt{A}[\mathbf{c}_j]\mathtt{L} \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ fails.

2. **Left of an application, *i.e.* $\mathtt{C} = \mathtt{C}_1\, s$ and $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$ and $h \neq \lambda$.** The only way to derive $r = \mathtt{C}_1[t]\, s$ is by having $\mathtt{C}_1[t] \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta \cup \mathcal{K}_\vartheta$. Moreover, t is a $\mathtt{C}$-reduction place, which implies that t is also a $\mathtt{C}_1$-reduction place (by Lem. 52). So we may apply the *i.h.* to conclude that this is impossible.

3. **Non-structural substitution, *i.e.* $\mathtt{C} = \mathtt{C}_1[x\backslash s]$ with $s \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ and $x \notin \vartheta$.** The only way to derive $r = \mathtt{C}_1[t][x\backslash s]$ is by having $\mathtt{C}_1[t] \in \mathcal{N}_\vartheta$.
   Moreover, t is a $\mathtt{C}$-reduction place, which implies that t is also a $\mathtt{C}_1$-reduction place (by Lem. 52). By *i.h.* this is not possible.

4. **Structural substitution, *i.e.* $\mathtt{C} = \mathtt{C}_1[x\backslash s]$ and $s \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** The only way to derive $r = \mathtt{C}_1[t][x\backslash s]$ is by having $\mathtt{C}_1[t] \in \mathcal{N}_{\vartheta\cup\{x\}}$.
   Moreover, t is a $\mathtt{C}$-reduction place, which implies that t is also a $\mathtt{C}_1$-reduction place (by Lem. 52). Thus we may apply the *i.h.* to conclude that this is impossible.

5. **Inside a substitution, *i.e.* $\mathtt{C} = \mathtt{C}_1[\![x]\!][x\backslash\mathtt{C}_1]$ and $\mathtt{C}_1 \in \mathcal{C}^{\cdot}_\vartheta$.** The only way to derive $r = \mathtt{C}_1[\![x]\!][x\backslash\mathtt{C}_1[t]]$ is by having $\mathtt{C}_1[\![x]\!] \in \mathcal{N}_\vartheta$. Note that $x \notin \mathtt{fz}^\vartheta(\mathtt{C}_1)$, since $x \notin \vartheta$ and x cannot be bound by $\mathtt{C}_1$ (by Barendregt's convention).
   Moreover, t is a $\mathtt{C}$-reduction place, which implies that t is also a $\mathtt{C}_1$-reduction place (by Lem. 52). So we may apply the *i.h.* to conclude that this is impossible.

6. **Right of a weak structure or weak error term, *i.e.* $\mathtt{C} = s\,\mathtt{C}_1$ and $s \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** The only way to derive $r = s\,\mathtt{C}_1[t]$ is by having $\mathtt{C}_1[t] \in \mathcal{N}_\vartheta$.
   Moreover, t is a $\mathtt{C}$-reduction place, which implies that t is also a $\mathtt{C}_1$-reduction place (by Lem. 52). So we may apply the *i.h.* to conclude that this is impossible.

7. **Right of a constructor answer, *i.e.* $\mathtt{C} = s\,\mathtt{C}_1$ and $s \in \mathcal{K}_\vartheta$.** The only way to derive $r = s\,\mathtt{C}_1[t]$ is by having $\mathtt{C}_1[t] \in \mathcal{N}_\vartheta$.
   Moreover, t is a $\mathtt{C}$-reduction place, which implies that t is also a $\mathtt{C}_1$-reduction place (by Lem. 52). So we may apply the *i.h.* to conclude that this is impossible.

8. **Under an abstraction, *i.e.* $\mathtt{C} = \lambda x.\mathtt{C}_1$.** The only way to derive $r = \lambda x.\mathtt{C}_1[t]$ is by having $\mathtt{C}_1[t] \in \mathcal{N}_{\vartheta \cup \{x\}}$.
   Moreover, t is a $\mathtt{C}$-reduction place, which implies that t is also a $\mathtt{C}_1$-reduction place (by Lem. 52). So we may apply the *i.h.* to conclude that this is impossible.

9. **In the branch of a case, $\mathtt{C} = \mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}_1, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n$ and $t \in \mathcal{N}_\vartheta$ and $t \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ and $t_k \in \mathcal{N}_{\vartheta\cup\bar{x_k}}$ for all $k < j$ and $\mathtt{C}'_1[\mathtt{C}_2] \in \mathcal{C}^{h''}_{\vartheta\cup\bar{x_i}}$ and $h = \cdot$.** The only way to derive $r = \mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}_1[t], \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n$ is by having $\mathtt{C}_1[t] \in \mathcal{N}_{\vartheta\cup\{\bar{x_j}\}}$.
   Moreover, t is a $\mathtt{C}$-reduction place, which implies that t is also a $\mathtt{C}_1$-reduction place (by Lem. 52). So we may apply the *i.h.* to conclude that this is impossible.

10. **In the condition of a case, $\mathtt{C} = \mathtt{case}\ \mathtt{C}_1\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{C}^{\cdot}_\vartheta$ and $\mathtt{C}_1[\mathtt{C}_2] \in \mathcal{C}^{h''}_\vartheta$ and $h'' \notin \{\mathbf{c}_i\}_{i\in I}$ or $h'' = \mathbf{c}_j \in \{\mathbf{c}_i\}_{i\in I}$ and $|\mathtt{A}(\mathtt{C}, y)| \neq |\bar{x_j}|$ and $h = \cdot$.** The only way to derive $r = \mathtt{case}\ \mathtt{C}_1[t]\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{C}^{\cdot}_\vartheta$ and $\mathtt{C}_1[\mathtt{C}_2] \in \mathcal{C}^{h''}_\vartheta$ is by having $\mathtt{C}_1[t] \in \mathcal{N}_{\vartheta\cup\{\bar{x_j}\}}$.
    Moreover, t is a $\mathtt{C}$-reduction place, which implies that t is also a $\mathtt{C}_1$-reduction place (by Lem. 52). So we may apply the *i.h.* to conclude that this is impossible.

□

Lemma 58 (Unique decomposition). *If $\mathtt{C}_1[t_1] = \mathtt{C}_2[t_2]$ with $C \in \mathcal{C}^h_\vartheta$ for some h, such that t_i is a $\mathtt{C}_i$-reduction place for $i \in \{1, 2\}$, then $(\mathtt{C}_1, t_1) = (\mathtt{C}_2, t_2)$.*

Proof. By induction on the derivation of $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$:

1. **eBox (root), $\mathtt{C}_1 = \Box$.** By cases on the definition that t_1 is a $\mathtt{C}_1$-reduction place:

   (a) **If t_1 is the redex pattern of a beta-step.** Suppose that $\mathtt{C}_2$ were not empty. Let $t_1 = (\lambda x.s)\mathtt{L}\,u$. Then $\mathtt{C}_2[t_2] = (\lambda x.s)\mathtt{L}\,u$. By Cor. 54 we have that $\mathtt{L}$ can be split as $\mathtt{L} = \mathtt{L}_1\mathtt{L}_2$ such that $\mathtt{C}_2 = \mathtt{L}_2\,u$. This means that $t_2 = (\lambda x.s)\mathtt{L}_1$, so t_2 cannot be a $\mathtt{C}_2$-reduction place, as it is neither an application nor a variable nor a fix-expression nor a case expression. Hence this case is impossible.

   (b) **If t_1 is a variable x contracted by an ls-step.** This case is not possible, as there is no substitution binding x.

   (c) **If t_1 is a free variable x such that $x \notin \mathtt{fz}^\vartheta(\mathtt{C}_1) = \vartheta$.** Immediate, as $\mathtt{C}_2 = \Box$ so $t_2 = x \notin \vartheta = \mathtt{fz}^\vartheta(\mathtt{C}_2)$.

   (d) **If t_1 is the redex pattern of a fix-step.** The result is immediate since $\mathtt{C}_2$ must be empty.

(e) **If t_1 is the redex pattern of a case-step.** Suppose that $\mathtt{C}_2$ were not empty. Let $t_1 = \mathtt{case}\ \mathtt{A}[\mathbf{c}_j]\mathtt{L}\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ and $|\mathtt{A}[\mathbf{c}_j]\mathtt{L}| = |\bar{x}_j|$ and $j \in I$. Then $\mathtt{C}_2[t_2] = \mathtt{case}\ \mathtt{A}[\mathbf{c}_j]\mathtt{L}\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$. However, by Lem. 56, $\mathtt{C}_2$ must be empty.

2. **eAppL (left of an application), *i.e.* $\mathtt{C}_1 = \mathtt{C}_{11}\,s_1$ with $\mathtt{C}_{11} \in \mathcal{C}^h_\vartheta$ and $h \neq \lambda$.** Then $\mathtt{C}_{11}\,s_1 = \mathtt{C}_2[t_2]$. By case analysis on the formation rules for $\mathtt{C}_2$:

   (a) **eBox, $\mathtt{C}_2 = \Box$.** Impossible (the symmetric situation was already considered in the base case).

   (b) **eAppL, *i.e.* $\mathtt{C}_2 = \mathtt{C}_{21}\,s$ with $\mathtt{C}_{21} \in \mathcal{C}^h_\vartheta$.** Then $\mathtt{C}_{11}[t_1] = \mathtt{C}_{21}[t_2]$. The contexts $\mathtt{C}_{11}$ and $\mathtt{C}_{21}$ are both in $\mathcal{C}^h_\vartheta$, and each t_i is a $\mathtt{C}_{i1}$-reduction place (by Lem. 52), so by *i.h.* we have $(\mathtt{C}_{11}, t_1) = (\mathtt{C}_{21}, t_2)$.

   (c) **eAppRStruct, *i.e.* $h = \cdot$, $\mathtt{C}_2 = s_2\,\mathtt{C}_{21}$ and $s_2 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ and $\mathtt{C}_{21} \in \mathcal{C}^{h'}_\vartheta$.** This implies that $s_2 = \mathtt{C}_{11}[t_1]$ where t_1 is a $\mathtt{C}_{11}$-reduction place. Terms in $\mathcal{N}_\vartheta$ such as s_2 are in normal form and hence cannot have a reduction place such as t_1 under an evaluation context such as $\mathtt{C}_{11}$. This last fact is a direct application of Lem. 57.

   (d) **eAppRCons, *i.e.* $\mathtt{C}_2 = s_2\,\mathtt{C}_{21}$ and $s_2 \in \mathcal{K}_\vartheta$ and $h = \mathsf{hc}(s_2)$ and $\mathtt{C}_{21} \in \mathcal{C}^{h'}_\vartheta$.** The same argument as in the previous subcase (*i.e.* eAppRStruct) applies.

3. **eSubsLNonStruct (non-structural substitution), *i.e.* $\mathtt{C}_1 = \mathtt{C}_{11}[x\backslash s_1]$ with $s_1 \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ and $x \notin \vartheta$ and $\mathtt{C}_{11} \in \mathcal{C}^h_\vartheta$.** By case analysis on the formation rules for $\mathtt{C}_2$:

   (a) **Empty, $\mathtt{C}_2 = \Box$.** Impossible (the symmetric situation was already considered in the base case).

   (b) **Non-structural substitution, *i.e.* $\mathtt{C}_2 = \mathtt{C}_{21}[x\backslash s_2]$ and $s_2 \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** Note that each t_i is a $\mathtt{C}_{i1}$-reduction place by Lem. 52. By the *i.h.* on $\mathtt{C}_1$ we have that $(\mathtt{C}_1, t_1) = (\mathtt{C}_2, t_2)$, so we conclude.

   (c) **Structural substitution, *i.e.* $\mathtt{C}_2 = \mathtt{C}_{21}[x\backslash s_2]$ and $s_2 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** This case is impossible, as the formation rule for $\mathtt{C}_1$ implies that $s_2 \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.

   (d) **Inside a substitution, *i.e.* $\mathtt{C}_2 = \mathtt{C}_{21}[\![x]\!][x\backslash \mathtt{C}_{22}]$ and $\mathtt{C}_{21} \in \mathcal{C}^h_\vartheta$ and $\mathtt{C}_{22} \in \mathcal{C}^{\cdot}_\vartheta$.** We claim that this case is impossible. Note that we have that $\mathtt{C}_{11}[t_1] = \mathtt{C}_{21}[\![x]\!]$, where t_1 is a $\mathtt{C}_{11}$-reduction place by virtue of Lem. 52. Moreover $x \notin \vartheta$, and x is not bound by $\mathtt{C}_{21}$ (by Barendregt's convention), so $x \notin \mathsf{fz}^\vartheta(\mathtt{C}_{21})$; these conditions imply that x is a $\mathtt{C}_{21}$-reduction place.
   This allows us to apply the *i.h.*, obtaining $(\mathtt{C}_{11}, x) = (\mathtt{C}_{21}, t_1)$. Since $t_1 = x$ is a $\mathtt{C}_1$-reduction place by hypothesis, and x is bound by $\mathtt{C}_1$, we conclude that it must be involved in an ls-step. This implies that the substitution $[x\backslash s_1]$ contains an answer, that is, $s_1 = v\mathtt{L}$. But from the formation rule of $\mathtt{C}_2$, we also know that $s = \mathtt{C}_{22}[t_2]$.
   So the situation is such that $\mathtt{C}_{22}[t_2] = v\mathtt{L}$. By the fact that non-abstraction answer evaluation contexts such as$\mathtt{C}_{22}$ do not go below answers (Lem. 53) we conclude that t_2 must be of the form $v\mathtt{L}_1$. This is a contradiction, as t_2 is a $\mathtt{C}_2$-reduction place, which means that it must be either an application or a variable or a fix or a case expression.

4. **eSubsLStruct (structural substitution), *i.e.* $\mathtt{C}_1 = \mathtt{C}_{11}[x\backslash s_1]$ and $\mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ and $s_1 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** By case analysis on the formation rules for $\mathtt{C}_2$:

(a) **Empty, $\mathtt{C}_2 = \Box$.** Impossible (the symmetric situation was already considered in the base case).

(b) **Non-structural substitution, *i.e.* $\mathtt{C}_2 = \mathtt{C}_{21}[x\backslash s_2]$ and $s_2 \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** Impossible since $s_1 = s_2$ and $s_1 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.

(c) **Structural substitution, *i.e.* $\mathtt{C}_2 = \mathtt{C}_{21}[x\backslash s_2]$ and $s_2 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** Then $s_1 = s_2$ and each t_i is a $\mathtt{C}_{i1}$-reduction place as a consequence of Lem. 52, so we may apply the *i.h.* to conclude $(\mathtt{C}_{11}, t_1) = (\mathtt{C}_{21}, t_2)$, as required.

(d) **Inside a substitution, *i.e.* $\mathtt{C}_2 = \mathtt{C}_{21}[\![x]\!][x\backslash \mathtt{C}_{22}]$.** Then we have that $s_1 = \mathtt{C}_{22}[t_2]$. Note that t_2 is a $\mathtt{C}_{21}$-reduction place by Lem. 52. This is impossible since s_1 is a strong normal form, and it may not have a reduction place under an evaluation context (Lem. 57).

5. **eSubsR (inside a substitution), *i.e.* $\mathtt{C}_1 = \mathtt{C}_{11}[\![x]\!][x\backslash \mathtt{C}_{12}]$ and $\mathtt{C}_{11} \in \mathcal{C}^h_\vartheta$ and $\mathtt{C}_{12} \in \mathcal{C}^{\cdot}_\vartheta$.** By case analysis on the formation rules for $\mathtt{C}_2$:

   (a) **Empty, $\mathtt{C}_2 = \Box$.** Impossible (the symmetric situation was already considered in the base case).

   (b) **Non-structural substitution, *i.e.* $\mathtt{C}_2 = \mathtt{C}_{21}[x\backslash s]$.** Impossible (the symmetric situation was already considered in the case in which $\mathtt{C}_1$ is built up with a non-structural substitution).

   (c) **Structural substitution, *i.e.* $\mathtt{C}_2 = \mathtt{C}_{21}[x\backslash s_2]$ and $s_2 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** Impossible (the symmetric situation was already considered in the case in which $\mathtt{C}_1$ is built up with a structural substitution).

   (d) **Inside a substitution, *i.e.* $\mathtt{C}_2 = \mathtt{C}_{21}[\![x]\!][x\backslash \mathtt{C}_{22}]$.** Then each t_i is a $\mathtt{C}_{i1}$-reduction place, as a consequence of Lem. 52. By applying the *i.h.* we obtain that $(\mathtt{C}_{12}, t_1) = (\mathtt{C}_{22}, t_2)$, as required.

6. **eAppRStruct (right of a weak structure or error term), *i.e.* $h = \cdot$, $\mathtt{C}_1 = s_1\,\mathtt{C}_{11}$ and $s_1 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ and $\mathtt{C}_{11} \in \mathcal{C}^{h'}_\vartheta$.** By case analysis on the formation rules for $\mathtt{C}_2$:

   (a) **Empty, $\mathtt{C}_2 = \Box$.** Impossible (the symmetric situation was already considered in the case in which $\mathtt{C}_1$ is empty).

   (b) **Left of an application, *i.e.* $\mathtt{C}_2 = \mathtt{C}_{21}\,s_2$.** Impossible (the symmetric situation was already considered in the case in which $\mathtt{C}_1$ goes to the left of an application).

   (c) **Right of a structure, *i.e.* $\mathtt{C}_2 = s_2\,\mathtt{C}_{21}$ and $s_2 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** Then each t_i is a $\mathtt{C}_{i1}$-reduction place, as a consequence of Lem. 52. By applying the *i.h.* we conclude that $(\mathtt{C}_{11}, t_1) = (\mathtt{C}_{21}, t_2)$, as required.

   (d) **Right of a constructor answer, *i.e.* $\mathtt{C}_2 = s_2\,\mathtt{C}_{21}$ and $s_2 \in \mathcal{K}_\vartheta$.** Not possible since $s_1 = s_2$ and $\mathcal{K}_\vartheta \cap (\mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta) = \varnothing$.

7. **eAppRCons (right of a constructor answer), *i.e.* $\mathtt{C}_1 = s_1\,\mathtt{C}_{11}$, $s_1 \in \mathcal{K}_\vartheta$, $h = \mathsf{hc}(s_1)$ and $\mathtt{C}_{11} \in \mathcal{C}^{h'}_\vartheta$.** By case analysis on the formation rules for $\mathtt{C}_2$:

   (a) **Empty, $\mathtt{C}_2 = \Box$.** Impossible (the symmetric situation was already considered in the case in which $\mathtt{C}_1$ is empty).

   (b) **Left of an application, *i.e.* $\mathtt{C}_2 = \mathtt{C}_{21}\,s$.** Impossible (the symmetric situation was already considered in the case in which $\mathtt{C}_1$ goes to the left of an application).

(c) **Right of a structure, *i.e.* $\mathtt{C}_2 = s_2\,\mathtt{C}_{21}$ and $s_2 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** Not possible since $s_1 = s_2$ and $\mathcal{K}_\vartheta \cap (\mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta) = \varnothing$.

(d) **Right of a constructor answer, *i.e.* $\mathtt{C}_2 = s_2\,\mathtt{C}_{21}$ and $s_2 \in \mathcal{K}_\vartheta$.** Then each t_i is a $\mathtt{C}_{i1}$-reduction place, as a consequence of Lem. 52. By applying the *i.h.* we conclude that $(\mathtt{C}_{11}, t_1) = (\mathtt{C}_{21}, t_2)$, as required.

8. **eLam (under an abstraction), *i.e.* $\mathtt{C}_1 = \lambda x.\mathtt{C}_{11}$, $h = \lambda$ and $\mathtt{C}_{11} \in \mathcal{C}^{h'}_{\vartheta\cup\{x\}}$.** Then $\mathtt{C}_2$ cannot be empty (the symmetric situation was already considered in the case in which $\mathtt{C}_1$ is empty), so $\mathtt{C}_2$ must be of the form $\lambda x.\mathtt{C}_{21}$. By Lem. 52 we know that each t_i must be a $\mathtt{C}_{i1}$-reduction place, so we may apply the *i.h.* to conclude that $(\mathtt{C}_{11}, t_1) = (\mathtt{C}_{21}, t_2)$, as required.

9. **eCase1 (branch of a case), *i.e.* $h = \cdot$, $\mathtt{C}_1 = \mathtt{case}\ u_1\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow s_1, \ldots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}_{11}, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow s_n$ and $u_1 \in \mathcal{N}_\vartheta$ and $u_1 \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow u_i)_{i\in I}$ and $s_k \in \mathcal{N}_{\vartheta\cup\bar{x_k}}$ for all $k < j$ and $\mathtt{C}_{11} \in \mathcal{C}^{h'}_{\vartheta\cup\bar{x_i}}$.** By case analysis on the formation rules for $\mathtt{C}_2$:

   (a) **eBox (empty), $\mathtt{C}_2 = \Box$.** Impossible (the symmetric situation was already considered in the case in which $\mathtt{C}_1$ is empty).

   (b) **eCase2 (condition of a case), *i.e.* $\mathtt{C}_2 = \mathtt{case}\ \mathtt{C}_{21}\ \mathtt{of}\ (\mathbf{c}_j\bar{x}_j \Rightarrow s'_j)_{i\in I} \in \mathcal{C}^{\cdot}_\vartheta$ and $\mathtt{C}_{21} \in \mathcal{C}^h_\vartheta$ and either $h \notin \{\mathbf{c}_j\}_{j\in J}$ or $h = \mathbf{c}_k \in \{\mathbf{c}_j\}_{j\in J}$ and $|\mathtt{A}(\mathtt{C}, y)| \neq |\bar{x_k}|$.** Then by Lem. 52 we know that t_2 must be a $\mathtt{C}_{21}$-reduction place. Thus $u_1 = \mathtt{C}_{21}[t_2]$. This is not possible by Lem. 57.

   (c) **eCase1 (branch of a case), *i.e.* $\mathtt{C}_2 = \mathtt{case}\ u_2\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow s'_1, \ldots, \mathbf{c}_{j'}\bar{x_{j'}} \Rightarrow \mathtt{C}_{11}, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow s'_n$ and $u_2 \in \mathcal{N}_\vartheta$ and $u_2 \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s'_i)_{i\in I}$ and $s'_k \in \mathcal{N}_{\vartheta\cup\bar{x}}$ for all $k < j'$ and $\mathtt{C}_{21} \in \mathcal{C}^{h'}_{\vartheta\cup\bar{x_i}}$.** Then $j = j'$ and by Lem. 52 we know that each t_i must be a $\mathtt{C}_{i1}$-reduction place, so we may apply the *i.h.* to conclude that $(\mathtt{C}_{11}, t_1) = (\mathtt{C}_{21}, t_2)$, as required.

10. **eCase2 (body of a case), *i.e.* $\mathtt{C}_1 = \mathtt{case}\ \mathtt{C}_{11}\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$, $h = \cdot$, $\mathtt{C}_{11} \in \mathcal{C}^{h'}_\vartheta$ and either $h' \notin \{\mathbf{c}_i\}_{i\in I}$ or $h' = \mathbf{c}_j \in \{\mathbf{c}_i\}_{i\in I}$ and $|\mathtt{A}(\mathtt{C}, y)| \neq |\bar{x_j}|$.** By case analysis on the formation rules for $\mathtt{C}_2$:

    (a) **eBox (empty), $\mathtt{C}_2 = \Box$.** Impossible (the symmetric situation was already considered in the case in which $\mathtt{C}_1$ is empty).

    (b) **eCase2 (condition of a case), *i.e.* $\mathtt{C}_2 = \mathtt{case}\ \mathtt{C}_{21}\ \mathtt{of}\ (\mathbf{c}_j\bar{x}_j \Rightarrow s'_j)_{i\in I} \in \mathcal{C}^{\cdot}_\vartheta$ and $\mathtt{C}_{21} \in \mathcal{C}^h_\vartheta$ and either $h \notin \{\mathbf{c}_j\}_{j\in J}$ or $h = \mathbf{c}_k \in \{\mathbf{c}_j\}_{j\in J}$ and $|\mathtt{A}(\mathtt{C}, y)| \neq |\bar{x_k}|$.** Impossible (the symmetric situation was already considered in the case eCase1).

    (c) **eCase1 (branch of a case), *i.e.* $\mathtt{C}_2 = \mathtt{case}\ u_2\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow s'_1, \ldots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}_{11}, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow s'_n$ and $u_2 \in \mathcal{N}_\vartheta$ and $u_2 \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s'_i)_{i\in I}$ and $s'_k \in \mathcal{N}_{\vartheta\cup\bar{x_k}}$ for all $k < j$ and $\mathtt{C}_{11} \in \mathcal{C}^{h'}_{\vartheta\cup\bar{x_i}}$.** Impossible (the symmetric situation was already considered in the case eCase1).

$\Box$

Figure 6 ϑ-normal forms of the strategy ($\mathbb{X} \in \{\mathcal{S}_\vartheta, \mathcal{L}_\vartheta, \mathcal{E}_\vartheta, \mathcal{K}_\vartheta\}$)

$$\frac{}{\mathbf{c} \in \mathcal{K}_\vartheta}\ \textsc{cNfCons} \qquad \frac{t \in \mathcal{K}_\vartheta \quad s \in \mathcal{N}_\vartheta}{t\,s \in \mathcal{K}_\vartheta}\ \textsc{cNfApp} \qquad \frac{x \in \vartheta}{x \in \mathcal{S}_\vartheta}\ \textsc{sNfVar} \qquad \frac{t \in \mathcal{S}_\vartheta \quad s \in \mathcal{N}_\vartheta}{t\,s \in \mathcal{S}_\vartheta}\ \textsc{sNfApp}$$

$$\frac{t \in \mathcal{K}_\vartheta \cup \mathcal{L}_\vartheta \cup \mathcal{S}_\vartheta \quad t \not\succ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i\in I} \quad (s_i \in \mathcal{N}_{\vartheta \cup \bar{x}_i})_{i \in I}}{\texttt{case}\ t\ \texttt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{E}_\vartheta}\ \textsc{eNfStrt}$$

$$\frac{t \in \mathcal{E}_\vartheta \quad s \in \mathcal{N}_\vartheta}{t\,s \in \mathcal{E}_\vartheta}\ \textsc{eNfApp} \qquad \frac{t \in \mathcal{E}_\vartheta \quad (s_i \in \mathcal{N}_{\vartheta \cup \bar{x}_i})_{i\in I}}{\texttt{case}\ t\ \texttt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{E}_\vartheta}\ \textsc{eNfCase} \qquad \frac{t \in \mathcal{N}_{\vartheta \cup \{x\}}}{\lambda x.t \in \mathcal{L}_\vartheta}\ \textsc{lNfLam}$$

$$\frac{t \in \mathbb{X}_{\vartheta \cup \{x\}} \quad s \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta \quad x \in \mathsf{ngv}(t)}{t[x\backslash s] \in \mathbb{X}_\vartheta}\ \textsc{nfSubNG} \qquad \frac{t \in \mathbb{X}_\vartheta \quad x \notin \mathsf{ngv}(t)}{t[x\backslash s] \in \mathbb{X}_\vartheta}\ \textsc{nfSubG}$$

$$\frac{t \in \mathcal{K}_\vartheta}{t \in \mathcal{N}_\vartheta}\ \textsc{nfCons} \qquad \frac{t \in \mathcal{S}_\vartheta}{t \in \mathcal{N}_\vartheta}\ \textsc{nfStruct} \qquad \frac{t \in \mathcal{L}_\vartheta}{t \in \mathcal{N}_\vartheta}\ \textsc{nfLam} \qquad \frac{t \in \mathcal{E}_\vartheta}{t \in \mathcal{N}_\vartheta}\ \textsc{nfError}$$

5.1 Normal Forms of the Strategy

We present an inductive characterization of the normal forms of $\rightarrowtail^\vartheta_{\mathtt{sh}}$. Since reduction in $\rightarrowtail^\vartheta_{\mathtt{sh}}$ is parameterized over a set of frozen variables ϑ, the normal forms too will be parameterized by this set. The set of **normal forms** over ϑ ($\mathcal{N}_\vartheta$) is comprised of the **constant normal forms** over ϑ ($\mathcal{K}_\vartheta$), the **structure normal forms** over ϑ ($\mathcal{S}_\vartheta$), the **error normal forms** over ϑ ($\mathcal{E}_\vartheta$) and the **lambda normal forms** over ϑ ($\mathcal{L}_\vartheta$). They are defined in Fig. 6 and are similar to the characterization of the $\rightarrow_{\mathtt{sh}}$-normal forms (Fig. 1) except that: 1) the set of frozen variables is tracked, 2) rule nfSub is refined into rules nfSubNG, and 3) a new rule nfSubG is added due to the absence of $\mathtt{gc}$ in $\rightarrowtail^\vartheta_{\mathtt{sh}}$. In rules nfSubNG and nfSubG, the symbol $\mathbb{X}$ represents either $\mathcal{S}_\vartheta$, $\mathcal{E}_\vartheta$, $\mathcal{L}_\vartheta$ or $\mathcal{K}_\vartheta$. Rule nfSubG helps capture terms such as $z[y\backslash x][x\backslash s]$. Note that $x \in \mathsf{fv}(z[y\backslash x])$ but this term is in normal form for any s. However, x is not really "reachable" from z, it is would in fact be erased if we had $\mathtt{gc}$. The notion of a variable being "reachable" in this sense is defined as follows:

Definition 59. *The set of reachable or, better still,* **non-garbage variables** *of a term t, denoted $\mathsf{ngv}(t)$, are defined below*[1]*, where $\bar{b}$ stands for $(\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i\in I}$.*

$$\begin{array}{rcl}
\mathsf{ngv}(x) & := & \{x\} \\
\mathsf{ngv}(\lambda x.t) & := & \mathsf{ngv}(t) \setminus \{x\} \\
\mathsf{ngv}(ts) & := & \mathsf{ngv}(t) \cup \mathsf{ngv}(s) \\
\mathsf{ngv}(\mathtt{fix}(x.t)) & := & \mathsf{ngv}(t) \setminus \{x\} \\
\mathsf{ngv}(\mathbf{c}) & := & \varnothing \\
\mathsf{ngv}(\mathtt{case}\ t\ \mathtt{of}\ \bar{b}) & := & \mathsf{ngv}(t) \cup \bigcup_{i \in 1..n} \mathsf{ngv}(s_i) \setminus \bar{x}_i \\
\mathsf{ngv}(t[x\backslash s]) & := & (\mathsf{ngv}(t) \setminus \{x\}) \cup \begin{cases} \mathsf{ngv}(s) & \textit{if } x \in \mathsf{ngv}(t) \\ \varnothing & \textit{otherwise} \end{cases}
\end{array}$$

[1]They may alternatively be characterized as $\mathsf{ngv}(t) = \mathsf{fv}(\downarrow_{\mathtt{gc}}(t))$, where $\downarrow_{\mathtt{gc}}(t)$ simply removes all garbage substitutions [BBBK17].

The next result below states that Fig. 6 indeed characterizes the normal forms of the strategy.

Lemma 60 (Normal Form Decomposition)**.**

- *$t \in \mathcal{K}_\vartheta \Rightarrow t = \mathtt{A}[\mathbf{c}]\mathtt{L}$*
- *$t \in \mathcal{S}_\vartheta \Rightarrow t = \mathtt{E}[x]$ for some weak context $\mathtt{E}$ and variable x.*
- *$t \in \mathcal{L}_\vartheta \Rightarrow t = (\lambda x.s)\mathtt{L}$ for some x, s and $\mathtt{L}$.*
- *$t \in \mathcal{E}_\vartheta \Rightarrow t = \mathtt{F}[\mathtt{case}\ s\ \mathtt{of}\ \bar{b}]$ for some error context $\mathtt{F}$ and s an answer or a weak structure and $s \not\succ \bar{b}$.*

Proof. By simultaneous induction on the derivation of $t \in \mathcal{K}_\vartheta$, $t \in \mathcal{S}_\vartheta$, $t \in \mathcal{E}_\vartheta$, and $t \in \mathcal{L}_\vartheta$.

- cNfCons. We set $\mathtt{A} := \Box$ and $\mathtt{L} := \epsilon$.
- cNfApp. The derivation ends in:

$$\frac{t_1 \in \mathcal{K}_\vartheta \quad t_2 \in \mathcal{N}_\vartheta}{t_1\, t_2 \in \mathcal{K}_\vartheta}\ \textsc{cNfApp}$$

  By the *i.h.* on $t_1 \in \mathcal{K}_\vartheta$, there exists $\mathtt{A}', \mathbf{c}'$ and $\mathtt{L}'$ s.t. $t_1 = \mathtt{A}'[\mathbf{c}']\mathtt{L}'$. We set $\mathtt{A} := \mathtt{A}'\mathtt{L}'\, t_2$, $\mathbf{c} := \mathbf{c}'$ and $\mathtt{L} := \epsilon$ and conclude.

- sNfVar. We set $\mathtt{E} = \Box$.
- sNfApp. The derivation ends in:

$$\frac{t_1 \in \mathcal{S}_\vartheta \quad t_2 \in \mathcal{N}_\vartheta}{t_1\, t_2 \in \mathcal{S}_\vartheta}\ \textsc{sNfApp}$$

  By the *i.h.* on $t_1 \in \mathcal{S}_\vartheta$ we have $t_1 = \mathtt{E}'[x]$. Therefore, we set $\mathtt{E} := \mathtt{E}'\, t_2$ and conclude.

- eNfStrt. The derivation ends in:

$$\frac{u \in \mathcal{K}_\vartheta \cup \mathcal{L}_\vartheta \cup \mathcal{S}_\vartheta \quad u \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \quad (s_i \in \mathcal{N}_{\vartheta\cup\bar{x}_i})_{i\in I}}{\mathtt{case}\ u\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{E}_\vartheta}\ \textsc{eNfStrt}$$

  By the *i.h.* on $u \in \mathcal{K}_\vartheta \cup \mathcal{L}_\vartheta \cup \mathcal{S}_\vartheta$, u is either an answer or a weak structure. Also, we have $u \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$. Therefore we set $\mathtt{F} := \Box$ and conclude.

- eNfApp. The derivation ends in:

$$\frac{t_1 \in \mathcal{E}_\vartheta \quad t_2 \in \mathcal{N}_\vartheta}{t_1\, t_2 \in \mathcal{E}_\vartheta}\ \textsc{eNfApp}$$

  By the *i.h.* on $t_1 \in \mathcal{E}_\vartheta$ we have $t_1 = \mathtt{F}'[\mathtt{case}\ s\ \mathtt{of}\ \bar{b}]$. Therefore, we set $\mathtt{F} := \mathtt{F}'\, t_2$ and conclude.

- eNfCase. The derivation ends in:

$$\frac{u \in \mathcal{E}_\vartheta \quad (s_i \in \mathcal{N}_{\vartheta\cup\bar{x}_i})_{i\in I}}{\mathtt{case}\ u\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{E}_\vartheta}\ \textsc{eNfCase}$$

  We apply the *i.h.* on $u \in \mathcal{E}_\vartheta$ to obtain $\mathtt{F}'$, then set $\mathtt{F} := \mathtt{case}\ \mathtt{F}'\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$.

- lNfLam. The derivation ends in:

$$\frac{t \in \mathcal{N}_{\vartheta\cup\{x\}}}{\lambda x.t \in \mathcal{L}_\vartheta}\ \textsc{lNfLam}$$

We set $\mathtt{L} := \epsilon$ and conclude.

- nfSub. The derivation ends in:

$$\frac{t_1 \in \mathbb{X}_{\vartheta\cup\{y\}} \quad t_2 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta \quad y \in \mathsf{ngv}(t_1)}{t_1[y\backslash t_2] \in \mathbb{X}_\vartheta}\ \textsc{nfSub}$$

We must consider each case: $\mathbb{X} \in \{\mathcal{S}, \mathcal{L}, \mathcal{K}, \mathcal{E}\}$.

  - If $\mathbb{X}$ is $\mathcal{S}$, by the *i.h.* on $t_1 \in \mathcal{S}_{\vartheta\cup\{y\}}$, we have $t_1 = \mathtt{E}'[x]$. We set $\mathtt{E} := \mathtt{E}'[y\backslash t_2]$ and conclude.
  - If $\mathbb{X}$ is $\mathcal{L}$, by the *i.h.* on $t_1 \in \mathcal{L}_{\vartheta\cup\{y\}}$, there exists x, s and $\mathtt{L}'$ s.t. $t_1 = (\lambda x.s)\mathtt{L}'$. We set $\mathtt{L} := \mathtt{L}'[y\backslash t_2]$ and conclude.
  - If $\mathbb{X}$ is $\mathcal{K}$, By the *i.h.* on $t_1 \in \mathcal{K}_{\vartheta\cup\{y\}}$, there exists $\mathtt{A}', \mathbf{c}'$ and $\mathtt{L}'$ s.t. $t_1 = \mathtt{A}'[\mathbf{c}']\mathtt{L}'$. We set $\mathtt{A} := \mathtt{A}'$, $\mathbf{c} := \mathbf{c}'$ and $\mathtt{L} := \mathtt{L}'[y\backslash t_2]$ and conclude.
  - If $\mathbb{X}$ is $\mathcal{E}$, by the *i.h.* on $t_1 \in \mathcal{E}_{\vartheta\cup\{y\}}$, we have $t_1 = \mathtt{F}'[\mathtt{case}\ s\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}]$, s an answer or a weak structure and $s \not\succ \bar{b}$. We set $\mathtt{F} := \mathtt{F}'[y\backslash t_2]$ and conclude.

- nfSubG. The derivation ends in:

$$\frac{t_1 \in \mathbb{X}_\vartheta \quad x \notin \mathsf{ngv}(t_1)}{t_1[x\backslash t_2] \in \mathbb{X}_\vartheta}\ \textsc{nfSubG}$$

We must consider each case: $\mathbb{X} \in \{\mathcal{S}, \mathcal{L}, \mathcal{K}, \mathcal{E}\}$.

  - If $\mathbb{X}$ is $\mathcal{S}$, by the *i.h.* on $t_1 \in \mathcal{S}_\vartheta$, we have $t_1 = \mathtt{E}'[x]$. We set $\mathtt{E} := \mathtt{E}'[x\backslash t_2]$ and conclude.
  - If $\mathbb{X}$ is $\mathcal{L}$, by the *i.h.* on $t_1 \in \mathcal{L}_\vartheta$, there exists x, s and $\mathtt{L}'$ s.t. $t_1 = (\lambda x.s)\mathtt{L}'$. We set $\mathtt{L} := \mathtt{L}'[x\backslash t_2]$ and conclude.
  - If $\mathbb{X}$ is $\mathcal{K}$, By the *i.h.* on $t_1 \in \mathcal{K}_\vartheta$, there exists $\mathtt{A}', \mathbf{c}'$ and $\mathtt{L}'$ s.t. $t_1 = \mathtt{A}'[\mathbf{c}']\mathtt{L}'$. We set $\mathtt{A} := \mathtt{A}'$, $\mathbf{c} := \mathbf{c}'$ and $\mathtt{L} := \mathtt{L}'[x\backslash t_2]$ and conclude.
  - If $\mathbb{X}$ is $\mathcal{E}$, by the *i.h.* on $t_1 \in \mathcal{E}_\vartheta$, we have $t_1 = \mathtt{F}'[\mathtt{case}\ s\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}]$, s an answer or a weak structure and $s \not\succ \bar{b}$. We set $\mathtt{F} := \mathtt{F}'[x\backslash t_2]$ and conclude.

$\square$

Lemma 61. *If $\mathcal{C} \in \mathcal{C}^h_\vartheta$, then $x \in \mathsf{ngv}(\mathcal{C}[\![x]\!])$.*

Proof. By induction on the derivation of $\mathcal{C}^h_\vartheta$

- eBox. The derivation ends in:

$$\frac{}{\Box \in \mathcal{C}^{\cdot}_\vartheta}\ \textsc{eBox}$$

Immediate.

- eLam. The derivation ends in:
$$\frac{\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta \cup \{y\}}}{\lambda y.\mathtt{C}_1 \in \mathcal{C}^\lambda_\vartheta}\ \textsc{eLam}$$
Then $\mathtt{C} = \lambda y.\mathtt{C}_1$ and by *i.h.* $x \in \mathsf{ngv}(\mathtt{C}_1[\![x]\!])$. Also, $\mathsf{ngv}(\lambda y.\mathtt{C}_1[\![x]\!]) = \mathsf{ngv}(\mathtt{C}_1[\![x]\!]) \setminus \{y\}$. Therefore, we conclude that $x \in \mathsf{ngv}(\lambda y.\mathtt{C}_1[\![x]\!])$ because x and y are distinct.

- eAppL. The derivation ends in:
$$\frac{\mathtt{C}_1 \in \mathcal{C}^h_\vartheta \quad h \neq \lambda}{\mathtt{C}_1\, t \in \mathcal{C}^h_\vartheta}\ \textsc{eApp-L}$$
Then $\mathtt{C} = \mathtt{C}_1\, t$ and by *i.h.* $x \in \mathsf{ngv}(\mathtt{C}_1[\![x]\!])$. Also, $\mathsf{ngv}(\mathtt{C}_1[\![x]\!]\, t) = \mathsf{ngv}(\mathtt{C}_1[\![x]\!]) \cup \mathsf{ngv}(t)$. Therefore, we conclude that $x \in \mathsf{ngv}(\mathtt{C}_1[\![x]\!]\, t)$.

- eAppRStruct. The derivation ends in:
$$\frac{t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta \quad \mathtt{C}_1 \in \mathcal{C}^h_\vartheta}{t\, \mathtt{C}_1 \in \mathcal{C}^{\cdot}_\vartheta}\ \textsc{eAppRStruct}$$
Then $\mathtt{C} = t\, \mathtt{C}_1$ and by *i.h.* $x \in \mathsf{ngv}(\mathtt{C}_1[\![x]\!])$. Also, $\mathsf{ngv}(t\, \mathtt{C}_1[\![x]\!]) = \mathsf{ngv}(t) \cup \mathsf{ngv}(\mathtt{C}_1[\![x]\!])$. Therefore, we conclude that $x \in \mathsf{ngv}(t\, \mathtt{C}_1[\![x]\!])$.

- eAppRCons. The derivation ends in:
$$\frac{t \in \mathcal{K}_\vartheta \quad \mathtt{C}_1 \in \mathcal{C}^h_\vartheta}{t\, \mathtt{C}_1 \in \mathcal{C}^{\mathsf{hc}(t)}_\vartheta}\ \textsc{eAppRCons}$$
Then $\mathtt{C} := t\, \mathtt{C}_1$ and by *i.h.* $x \in \mathsf{ngv}(\mathtt{C}_1[\![x]\!])$. Also, $\mathsf{ngv}(t\, \mathtt{C}_1[\![x]\!]) = \mathsf{ngv}(t) \cup \mathsf{ngv}(\mathtt{C}_1[\![x]\!])$. Therefore, we conclude that $x \in \mathsf{ngv}(t\, \mathtt{C}_1[\![x]\!])$.

- eSubsLNonStruct. The derivation ends in:
$$\frac{\mathtt{C}_1 \in \mathcal{C}^h_\vartheta \quad t \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta \quad y \notin \vartheta}{\mathtt{C}_1[y\backslash t] \in \mathcal{C}^h_\vartheta}\ \textsc{eSubsLNonStruct}$$
Then $\mathtt{C} := \mathtt{C}_1[y\backslash t]$ and by *i.h.* $x \in \mathsf{ngv}(\mathtt{C}_1[\![x]\!])$.
Also, $\mathsf{ngv}(\mathtt{C}_1[\![x]\!][y\backslash t]) = \mathsf{ngv}(\mathtt{C}_1[\![x]\!]) \cup \begin{cases} \mathsf{ngv}(t) & \text{if } y \in \mathsf{ngv}(\mathtt{C}_1[\![x]\!]) \\ \varnothing & \text{otherwise} \end{cases}$
In either case, we conclude that $x \in \mathsf{ngv}(\mathtt{C}_1[\![x]\!][y\backslash t])$.

- eSubsLStruct. The derivation ends in:
$$\frac{\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta \cup \{y\}} \quad t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta}{\mathtt{C}_1[y\backslash t] \in \mathcal{C}^h_\vartheta}\ \textsc{eSubsLStruct}$$
Then $\mathtt{C} = \mathtt{C}_1[y\backslash t]$ and , by *i.h.* $x \in \mathsf{ngv}(\mathtt{C}_1[\![x]\!])$.
Also, $\mathsf{ngv}(\mathtt{C}_1[\![x]\!][y\backslash t]) = \mathsf{ngv}(\mathtt{C}_1[\![x]\!]) \cup \begin{cases} \mathsf{ngv}(t) & \text{if } y \in \mathsf{ngv}(\mathtt{C}_1[\![x]\!]) \\ \varnothing & \text{otherwise} \end{cases}$
In either case, we conclude that $x \in \mathsf{ngv}(\mathtt{C}_1[\![x]\!][y\backslash t])$.

- eSubsR. The derivation ends in:

$$\frac{\mathtt{C}_1 \in \mathcal{C}^h_\vartheta \quad \mathtt{C}_2 \in \mathcal{C}^\cdot_\vartheta}{\mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}_2] \in \mathcal{C}^h_\vartheta} \text{ eSubs-R}$$

  Then $\mathtt{C} = \mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}_2]$ and by *i.h.* $y \in \mathsf{ngv}(\mathtt{C}_1[\![y]\!])$ and $x \in \mathsf{ngv}(\mathtt{C}_2[\![x]\!])$.
  Also, $\mathsf{ngv}(\mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}_2[\![x]\!]]) = \mathsf{ngv}(\mathtt{C}_1[\![y]\!]) \setminus \{y\} \cup \mathsf{ngv}(\mathtt{C}_2[\![x]\!])$.
  Therefore, we conclude that $x \in \mathsf{ngv}(\mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}_2[\![x]\!]])$.

- eCase-1. The derivation ends in:

$$\frac{t \in \mathcal{N}_\vartheta \quad t \not> (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \quad t_j \in \mathcal{N}_{\vartheta\cup\bar{x}_j} \text{ for all } j < i \quad \mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\bar{x}_i}}{\mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x}_1 \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x}_j \Rightarrow \mathtt{C}_1, \ldots, \mathbf{c}_n\bar{x}_n \Rightarrow t_n \in \mathcal{C}^\cdot_\vartheta} \text{ eCase-3}$$

  Then $\mathtt{C} = \mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x}_1 \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x}_j \Rightarrow \mathtt{C}_1, \ldots, \mathbf{c}_n\bar{x}_n \Rightarrow t_n$ and by *i.h.* $x \in \mathsf{ngv}(\mathtt{C}_1[\![x]\!])$.
  Also, $x \notin \bar{x}_j$. Also, $\mathsf{ngv}(\mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x}_1 \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x}_j \Rightarrow \mathtt{C}_1[\![x]\!], \ldots, \mathbf{c}_n\bar{x}_n \Rightarrow t_n) = \mathsf{ngv}(t) \cup \mathsf{ngv}(\mathtt{C}_1[\![x]\!]) \setminus \bar{x}_j \cup \bigcup_{i\in 2..n} \mathsf{ngv}(\mathbf{c}_i\bar{x}_i) \setminus \bar{x}_i$.
  Therefore, we conclude that $x \in \mathsf{ngv}(\mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x}_1 \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x}_j \Rightarrow \mathtt{C}_1[\![x]\!], \ldots, \mathbf{c}_n\bar{x}_n \Rightarrow t_n)$.

- eCase-2. The derivation ends in:

$$\frac{\mathtt{C} \in \mathcal{C}^h_\vartheta \quad h \notin \{\mathbf{c}_i\}_{i\in I} \text{ or } h = \mathbf{c}_j \in \{\mathbf{c}_i\}_{i\in I} \text{ and } |\mathtt{A}(\mathtt{C}_1, y)| \neq |\bar{x}_j|}{\mathtt{case}\ \mathtt{C}_1\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{C}^\cdot_\vartheta} \text{ eCase1}$$

  Then $\mathtt{C} = \mathtt{case}\ \mathtt{C}_1\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ and by *i.h.* $x \in \mathsf{ngv}(\mathtt{C}_1[\![x]\!])$. Also, $\mathsf{ngv}(\mathtt{case}\ \mathtt{C}_1[\![x]\!]\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}) = \mathsf{ngv}(\mathtt{C}_1[\![x]\!]) \cup \bigcup_{i\in 1..n} \mathsf{ngv}(s_i) \setminus \bar{x}_i$.
  Therefore, we conclude that $x \in \mathsf{ngv}(\mathtt{case}\ \mathtt{C}_1[\![x]\!]\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I})$.

□

Lemma 62 (Strengthening for normal-form judgements)**.** *$t \in \mathcal{N}_{\vartheta\cup\{x\}}$ and $x \notin \mathsf{ngv}(t) \Rightarrow t \in \mathcal{N}_\vartheta$. Similarly for $t \in \mathcal{K}_\vartheta$, $t \in \mathcal{S}_\vartheta$, $t \in \mathcal{E}_\vartheta$, and $t \in \mathcal{L}_\vartheta$.*

Proof. By simultaneous induction on the derivation of $t \in \mathcal{K}_\vartheta$, $t \in \mathcal{S}_\vartheta$, $t \in \mathcal{E}_\vartheta$, and $t \in \mathcal{L}_\vartheta$.

- sNfVar. The derivation ends in:

$$\frac{y \in \vartheta \cup \{x\}}{y \in \mathcal{S}_{\vartheta\cup\{x\}}} \text{ sNfVar}$$

  Since $x \notin \mathsf{ngv}(y)$, $x \neq y$. Thus, $y \in \vartheta$ and we conclude from sNfVar:

$$\frac{y \in \vartheta}{y \in \mathcal{S}_\vartheta} \text{ sNfVar}$$

- cNfCons. The derivation ends in:

$$\frac{t \in \mathcal{K}_{\vartheta\cup\{x\}}}{t \in \mathcal{N}_{\vartheta\cup\{x\}}} \text{ nfCons}$$

  We conclude from the *i.h.*.

- cNfApp. The derivation ends in:

$$\frac{t_1 \in \mathcal{K}_{\vartheta\cup\{x\}} \quad t_2 \in \mathcal{N}_{\vartheta\cup\{x\}}}{t_1\, t_2 \in \mathcal{K}_{\vartheta\cup\{x\}}}\ \textsc{cNfApp}$$

We conclude from the *i.h.* on $t_1 \in \mathcal{K}_{\vartheta\cup\{x\}}$ and $t_2 \in \mathcal{N}_{\vartheta\cup\{x\}}$.

- sNfApp. The derivation ends in:

$$\frac{t_1 \in \mathcal{S}_{\vartheta\cup\{x\}} \quad t_2 \in \mathcal{N}_{\vartheta\cup\{x\}}}{t_1\, t_2 \in \mathcal{S}_{\vartheta\cup\{x\}}}\ \textsc{sNfApp}$$

Similar to previous case.

- eNfApp. The derivation ends in:

$$\frac{t_1 \in \mathcal{E}_{\vartheta\cup\{x\}} \quad t_2 \in \mathcal{N}_{\vartheta\cup\{x\}}}{t_1\, t_2 \in \mathcal{E}_{\vartheta\cup\{x\}}}\ \textsc{eNfApp}$$

Similar to previous case.

- eNfStrt. The derivation ends in:

$$\frac{u \in \mathcal{K}_{\vartheta\cup\{x\}} \cup \mathcal{L}_{\vartheta\cup\{x\}} \cup \mathcal{S}_{\vartheta\cup\{x\}} \quad u \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \quad (s_i \in \mathcal{N}_{\vartheta\cup\{x\}\cup\bar{x}_i})_{i\in I}}{\texttt{case}\ u\ \texttt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{E}_{\vartheta\cup\{x\}}}\ \textsc{eNfStrt}$$

We apply the *i.h.* on $u \in \mathcal{K}_{\vartheta\cup\{x\}} \cup \mathcal{L}_{\vartheta\cup\{x\}} \cup \mathcal{S}_{\vartheta\cup\{x\}}$ and $(s_i \in \mathcal{N}_{\vartheta\cup\{x\}\cup\bar{x}_i})_{i\in I}$ and conclude immediately from eNfStrt.

- eNfCase. The derivation ends in:

$$\frac{u \in \mathcal{E}_{\vartheta\cup\{x\}} \quad (s_i \in \mathcal{N}_{\vartheta\cup\{x\}\cup\bar{x}_i})_{i\in I}}{\texttt{case}\ u\ \texttt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{E}_{\vartheta\cup\{x\}}}\ \textsc{eNfCase}$$

We conclude by applying the *i.h.* on $u \in \mathcal{E}_{\vartheta\cup\{x\}}$ and $(s_i \in \mathcal{N}_{\vartheta\cup\{x\}\cup\bar{x}_i})_{i\in I}$.

- lNfLam. The derivation ends in:

$$\frac{t \in \mathcal{N}_{\vartheta\cup\{x\}\cup\{y\}}}{\lambda y.t \in \mathcal{L}_{\vartheta\cup\{x\}}}\ \textsc{lNfLam}$$

We may assume w.l.o.g that $y \neq x$. Therefore, $x \notin \mathsf{ngv}(\lambda y.t)$, implies $x \notin \mathsf{ngv}(t)$. Therefore we conclude from the *i.h.* and an application of lNfLam.

- nfSub. The derivation ends in:

$$\frac{t_1 \in \mathbb{X}_{\vartheta\cup\{x\}\cup\{y\}} \quad t_2 \in \mathcal{S}_{\vartheta\cup\{x\}} \cup \mathcal{E}_{\vartheta\cup\{x\}} \quad y \in \mathsf{ngv}(t_1)}{t_1[y\backslash t_2] \in \mathbb{X}_{\vartheta\cup\{x\}}}\ \textsc{nfSub}$$

We may assume w.l.o.g that $y \neq x$. Since $y \in \mathsf{ngv}(t_1)$, we have $\mathsf{ngv}(t_1[y\backslash t_2]) = \mathsf{ngv}(t_1) \setminus \{y\} \cup \mathsf{ngv}(t_2)$. Therefore, $x \notin \mathsf{ngv}(t_1[y\backslash t_2])$, implies $x \notin \mathsf{ngv}(t_1)$ and $x \notin \mathsf{ngv}(t_2)$. Therefore we conclude from the *i.h.* on $t_1 \in \mathbb{X}_{\vartheta\cup\{x\}\cup\{y\}}$ and $t_2 \in \mathcal{S}_{\vartheta\cup\{x\}} \cup \mathcal{E}_{\vartheta\cup\{x\}}$ and an application of nfSub.

- NFSUBG. The derivation ends in:

$$\frac{t_1 \in \mathbb{X}_{\vartheta \cup \{x\}} \quad y \notin \mathsf{ngv}(t_1)}{t_1[y \backslash t_2] \in \mathbb{X}_{\vartheta \cup \{x\}}} \text{ NFSUBG}$$

Since $y \notin \mathsf{ngv}(t_1)$, then $\mathsf{ngv}(t_1[y \backslash t_2]) = \mathsf{ngv}(t_1)$. Therefore, $x \notin \mathsf{ngv}(t)$ implies $x \notin \mathsf{ngv}(t_1)$. Thus we may apply the *i.h.* to $t_1 \in \mathbb{X}_{\vartheta \cup \{x\}}$, obtaining $t_1 \in \mathbb{X}_\vartheta$. We then conclude by applying NFSUBG.

□

Lemma 63 (Weakening for normal-form judgements)**.** *$t \in \mathcal{N}_\vartheta$ and $x \notin \vartheta$ and x not bound in t, then $t \in \mathcal{N}_{\vartheta \cup \{x\}}$. The same holds for $\mathcal{S}_\vartheta$, $\mathcal{L}_\vartheta$, $\mathcal{K}_\vartheta$ and $\mathcal{E}_\vartheta$.*

Proof. By simultaneous induction on $t \in \mathcal{N}_\vartheta$, $\mathcal{S}_\vartheta$, $\mathcal{L}_\vartheta$, $\mathcal{K}_\vartheta$ and $\mathcal{E}_\vartheta$. □

Lemma 64 (Weakening for contexts)**.** *$\mathtt{C} \in \mathcal{C}^h_\vartheta$ and $x \notin \vartheta$ and x not bound in $\mathtt{C}$, then $\mathtt{C} \in \mathcal{C}^h_{\vartheta \cup \{x\}}$.*

Proof. By induction on $\mathtt{C} \in \mathcal{C}^h_\vartheta$.

□

Lemma 65 (Normal-forms)**.** $\mathcal{N}_\vartheta \subseteq \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$.

Proof. By simultaneous induction on the derivation of $t \in \mathcal{K}_\vartheta$, $t \in \mathcal{S}_\vartheta$, $t \in \mathcal{E}_\vartheta$, and $t \in \mathcal{L}_\vartheta$.

- CNFCONS. Immediate.
- CNFAPP. The derivation ends in:

$$\frac{t_1 \in \mathcal{K}_\vartheta \quad t_2 \in \mathcal{N}_\vartheta}{t_1\, t_2 \in \mathcal{K}_\vartheta} \text{ CNFAPP}$$

By *i.h.* on $t_1 \in \mathcal{K}_\vartheta$ and $t_2 \in \mathcal{N}_\vartheta$, we have $t_1 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$ and $t_2 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$. From Lem. 60 we know $t_1 \in \mathcal{K}_\vartheta \Rightarrow t_1 = \mathtt{A}[\mathbf{c}]\mathtt{L}$, therefore t_1 is a constant answer and not an abstraction answer and we can conclude that $t_1 t_2 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$.

- SNFVAR. Immediate.
- SNFAPP. The derivation ends in:

$$\frac{t_1 \in \mathcal{S}_\vartheta \quad t_2 \in \mathcal{N}_\vartheta}{t_1\, t_2 \in \mathcal{S}_\vartheta} \text{ SNFAPP}$$

By *i.h.* on $t_1 \in \mathcal{S}_\vartheta$ and $t_2 \in \mathcal{N}_\vartheta$, we have $t_1 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$ and $t_2 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$. From Lem. 60 we know $t_1 \in \mathcal{S}_\vartheta \Rightarrow t_1 = \mathtt{E}[x]$, therefore we know t_1 is not an abstraction answer and we can conclude that $t_1 t_2 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$.

- ENFAPP. The derivation ends in:

$$\frac{t_1 \in \mathcal{E}_\vartheta \quad t_2 \in \mathcal{N}_\vartheta}{t_1\, t_2 \in \mathcal{E}_\vartheta} \text{ ENFAPP}$$

By *i.h.* on $t_1 \in \mathcal{E}_\vartheta$ and $t_2 \in \mathcal{N}_\vartheta$, we have $t_1 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$ and $t_2 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$. From Lem. 60 we know $t_1 \in \mathcal{E}_\vartheta \Rightarrow t_1 = \mathtt{F}[\mathtt{case}\ s\ \mathtt{of}\ \bar{b}]$, therefore we know t_1 is not an abstraction answer and we can conclude that $t_1 t_2 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$.

- eNfStrt. The derivation ends in:

$$\frac{u \in \mathcal{K}_\vartheta \cup \mathcal{L}_\vartheta \cup \mathcal{S}_\vartheta \quad u \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \quad (s_i \in \mathcal{N}_{\vartheta\cup\bar{x}_i})_{i\in I}}{\texttt{case}\ u\ \texttt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{E}_\vartheta}\ \textsc{eNfStrt}$$

By *i.h.* on $u \in \mathcal{K}_\vartheta \cup \mathcal{L}_\vartheta \cup \mathcal{S}_\vartheta$ we have $u \in \mathsf{NF}(\rightarrowtail^\vartheta_{\texttt{sh}})$. We can apply the *i.h.* for each $(s_i \in \mathcal{N}_{\vartheta\cup\bar{x}_i})_{i\in I}$, giving us $(s_i \in \mathsf{NF}(\rightarrowtail^\vartheta_{\texttt{sh}}))_{i\in I}$. Therefore, we can conclude $\texttt{case}\ u\ \texttt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathsf{NF}(\rightarrowtail^\vartheta_{\texttt{sh}})$ because we cannot apply $\rightarrow_{\texttt{case}}$ since $u \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$.

- eNfCase. The derivation ends in:

$$\frac{u \in \mathcal{E}_\vartheta \quad (s_i \in \mathcal{N}_{\vartheta\cup\bar{x}_i})_{i\in I}}{\texttt{case}\ u\ \texttt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{E}_\vartheta}\ \textsc{eNfCase}$$

By *i.h.* on $u \in \mathcal{E}_\vartheta$ we have $u \in \mathsf{NF}(\rightarrowtail^\vartheta_{\texttt{sh}})$. From Lem. 60 we know $u \in \mathcal{E}_\vartheta \Rightarrow u = \mathtt{F}[\texttt{case}\ s\ \texttt{of}\ \bar{b}]$ and hence is not a constant answer. We can apply the *i.h.* for each $(s_i \in \mathcal{N}_{\vartheta\cup\bar{x}_i})_{i\in I}$, giving us $(s_i \in \mathsf{NF}(\rightarrowtail^\vartheta_{\texttt{sh}}))_{i\in I}$. Therefore, we can conclude $\texttt{case}\ u\ \texttt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathsf{NF}(\rightarrowtail^\vartheta_{\texttt{sh}})$.

- lNfLam. The derivation ends in:

$$\frac{t_1 \in \mathcal{N}_{\vartheta\cup\{x\}}}{\lambda x.t_1 \in \mathcal{L}_\vartheta}\ \textsc{lNfLam}$$

By *i.h.* on $t_1 \in \mathcal{N}_{\vartheta\cup\{x\}}$, we have $t_1 \in \mathsf{NF}(\rightarrowtail^{\vartheta\cup\{x\}}_{\texttt{sh}})$, then $\lambda x.t_1 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\texttt{sh}})$ is immediate.

- nfSub. The derivation ends in:

$$\frac{t_1 \in \mathbb{X}_{\vartheta\cup\{x\}} \quad t_2 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta \quad x \in \mathsf{ngv}(t_1)}{t_1[x\backslash t_2] \in \mathbb{X}_\vartheta}\ \textsc{nfSub}$$

By *i.h.* on $t_1 \in \mathbb{X}_{\vartheta\cup\{x\}}$ and $t_2 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$, we have $t_1 \in \mathsf{NF}(\rightarrowtail^{\vartheta\cup\{x\}}_{\texttt{sh}})$ and $t_2 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\texttt{sh}})$. From Lem. 60 we know t_2 is either a weak structure or a weak error term, t_2 is not an answer. We can conclude $t_1[x\backslash t_2] \in \mathsf{NF}(\rightarrowtail^\vartheta_{\texttt{sh}})$ because we cannot apply $\texttt{lsv}$.

- nfSubG. The derivation ends in:

$$\frac{t_1 \in \mathbb{X}_\vartheta \quad x \notin \mathsf{ngv}(t_1)}{t_1[x\backslash t_2] \in \mathbb{X}_\vartheta}\ \textsc{nfSubG}$$

By *i.h.* on $t_1 \in \mathbb{X}_\vartheta$, we have $t_1 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\texttt{sh}})$. From Lem. 61, if we take the contrapositive we have $\mathtt{C} \notin \mathcal{C}^h_\vartheta$, for $x \notin \mathsf{ngv}(\mathtt{C}[\![x]\!])$. Therefore, we can conclude that $t_1[x\backslash t_2] \in \mathsf{NF}(\rightarrowtail^\vartheta_{\texttt{sh}})$.

□

Lemma 66 (Normal-forms)**.** *Suppose* $t \in \mathsf{NF}(\rightarrowtail^\vartheta_{\texttt{sh}})$.

1. *Then* $t \in \mathcal{N}_\vartheta$
2. *If* $x \in \mathsf{ngv}(t)$*, then* $\exists \mathtt{C}, h$ *s.t.* $t = \mathtt{C}[\![x]\!]$*, where* $\mathtt{C} \in \mathcal{C}^h_\vartheta$ *and* $x \notin \vartheta$

Proof. We show both items by simultaneous induction on t. Note that for the first item, due to nfStruct, nfLam, nfError and nfCons, it suffices to show that $t \in \mathcal{L}_\vartheta \cup \mathcal{K}_\vartheta \cup \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.

- $t = x$.
  - For item 1, let ϑ be any set of variables such that $\mathsf{fv}(x) \subseteq \vartheta$. By definition of $\mathsf{fv}()$, $x \in \vartheta$, and from
  $$\frac{x \in \vartheta}{x \in \mathcal{S}_\vartheta}\ \text{sNfVar}$$
  we have $x \in \mathcal{N}_\vartheta$.
  - For item 2, we set $\mathtt{C} := \Box$ from eBox and conclude.
- $t = \lambda y.s$.
  - For item 1, let ϑ be any set of variables such that $\mathsf{fv}(\lambda y.s) \subseteq \vartheta$. By the *i.h.* on s, $s \in \mathcal{N}_{\vartheta \cup \{y\}}$ since $\mathsf{fv}(s) \subseteq \vartheta \cup \{y\}$. Then from
  $$\frac{s \in \mathcal{N}_{\vartheta \cup \{y\}}}{\lambda x.s \in \mathcal{L}_\vartheta}\ \text{lNfLam}$$
  we have $\lambda y.s \in \mathcal{N}_\vartheta$.
  - For item 2, $x \neq y$, from the *i.h.* on s, $s = \mathtt{C}_1[\![x]\!]$, where $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$, for some h, and $x \notin \vartheta$. If $\vartheta = \vartheta' \cup \{y\}$, we apply:
  $$\frac{\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta' \cup \{y\}}}{\lambda y.\mathtt{C}_1 \in \mathcal{C}^\lambda_{\vartheta'}}\ \text{eLam}$$
  and conclude. Otherwise, we first apply Lem. 64 and then use eLam.
- $t = t_1\, t_2$.
  - For item 1, let ϑ be any set of variables such that $\mathsf{fv}(t_1 t_2) \subseteq \vartheta$. By the *i.h.*, we have $t_1 \in \mathcal{N}_\vartheta$ since $\mathsf{fv}(t_1) \subseteq \vartheta$ and $t_2 \in \mathcal{N}_\vartheta$ since $\mathsf{fv}(t_2) \subseteq \vartheta$. Then from $t_1 \in \mathcal{N}_\vartheta$, it must be the case that $t_1 \in \mathcal{L}_\vartheta \cup \mathcal{S}_\vartheta \cup \mathcal{K}_\vartheta \cup \mathcal{E}_\vartheta$. If $t_1 \in \mathcal{S}_\vartheta \cup \mathcal{K}_\vartheta \cup \mathcal{E}_\vartheta$, then we reason as follows:
  $$\frac{t_1 \in \mathcal{K}_\vartheta \quad t_2 \in \mathcal{N}_\vartheta}{t_1\, t_2 \in \mathcal{K}_\vartheta}\ \text{cNfApp} \qquad \frac{t_1 \in \mathcal{S}_\vartheta \quad t_2 \in \mathcal{N}_\vartheta}{t_1\, t_2 \in \mathcal{S}_\vartheta}\ \text{sNfApp} \qquad \frac{t_1 \in \mathcal{E}_\vartheta \quad t_2 \in \mathcal{N}_\vartheta}{t_1\, t_2 \in \mathcal{E}_\vartheta}\ \text{eNfApp}$$
  and conclude by applying nfCons, nfStruct and nfError, resp. Suppose $t_1 \in \mathcal{L}_\vartheta$. By Lem. 60 $t_1 = (\lambda x.s)\mathtt{L}$ and we could apply $\mathtt{lsv}$, contradicting $t_1 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$. This concludes the case.
  - For item 2, we consider two cases.
    * $x \in \mathsf{ngv}(t_1)$. Since $t_1 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$ by the *i.h.* w.r.t. item 1, and $x \in \mathsf{ngv}(t_1)$, then by the *i.h.* w.r.t. item 2, $\exists \mathtt{C}_1, h_1$ s.t. $t_1 = \mathtt{C}_1[\![x]\!]$, where $\mathtt{C}_1 \in \mathcal{C}^{h_1}_\vartheta$ and $x \notin \vartheta$. Moreover, we have $h \neq \lambda$, since $h = \lambda$ implies, by Lem. 45, that $\mathtt{C}[\![x]\!]$ is an abstraction answer, contradicting the hypothesis that $t_1 t_2 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$. Therefore we conclude by applying eApp-L.
    $$\frac{\mathtt{C}_1 \in \mathcal{C}^h_\vartheta \quad h \neq \lambda}{\mathtt{C}_1\, t_2 \in \mathcal{C}^h_\vartheta}\ \text{eApp-L}$$

* $x \in \mathsf{ngv}(t_2)$. Since $t_2 \in \mathsf{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$ by the *i.h.* w.r.t. item 1 and $x \in \mathsf{ngv}(t_2)$, then by the *i.h.* w.r.t. item 2, $\exists \mathtt{C}_1, h_1$ s.t. $t_2 = \mathtt{C}_1[\![x]\!]$, where $\mathtt{C}_1 \in \mathcal{C}^{h_1}_{\vartheta}$ and $x \notin \vartheta$. Additionally, by the *i.h.* w.r.t. item 1, $t_1 \in \mathsf{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$. Moreover, $t_1 \in \mathcal{S}_\vartheta \cup \mathcal{K}_\vartheta \cup \mathcal{E}_\vartheta$ since if we suppose $t_1 \in \mathcal{L}_\vartheta$, by Lem. 60 $t_1 = (\lambda x.s)\mathtt{L}$ and we could apply $\mathtt{lsv}$, contradicting $t_1 \in \mathsf{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$. In the case that $t_1 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$, we apply eAppRStruct and conclude. Otherwise, in the case that $t_1 \in \mathcal{K}_\vartheta$, we apply eAppRCons and conclude.

$$\frac{t_1 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta \quad \mathtt{C}_1 \in \mathcal{C}^{h}_{\vartheta}}{t_1\,\mathtt{C}_1 \in \mathcal{C}^{\cdot}_{\vartheta}}\ \textsc{eAppRStruct} \qquad \frac{t_1 \in \mathcal{K}_\vartheta \quad \mathtt{C}_1 \in \mathcal{C}^{h}_{\vartheta}}{t_1\,\mathtt{C}_1 \in \mathcal{C}^{\mathsf{hc}(t)}_{\vartheta}}\ \textsc{eAppRCons}$$

- $t = \mathtt{case}\ s\ \mathtt{of}\ \bar{b}$.
  - Let ϑ be any set of variables such that $\mathsf{fv}(\mathtt{case}\ s\ \mathtt{of}\ \bar{b}) \subseteq \vartheta$. By the *i.h.*, we have $s \in \mathcal{N}_\vartheta$ since $\mathsf{fv}(s) \subseteq \vartheta$. Likewise, we have $(s_i \in \mathcal{N}_{\vartheta \cup \{\bar{x_i}\}})_{i \in I}$ since $(\mathsf{fv}(s_i) \subseteq \vartheta \cup \{\bar{x_i}\})_{i \in I}$. Then from $s \in \mathcal{N}_\vartheta$, it must be the case that $s \in \mathcal{L}_\vartheta \cup \mathcal{S}_\vartheta \cup \mathcal{K}_\vartheta \cup \mathcal{E}_\vartheta$. If $s \in \mathcal{E}_\vartheta$, then we use:

$$\frac{s \in \mathcal{E}_\vartheta \quad (s_i \in \mathcal{N}_{\vartheta \cup \bar{x_i}})_{i \in I}}{\mathtt{case}\ s\ \mathtt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \in \mathcal{E}_\vartheta}\ \textsc{eNfCase}$$

    Otherwise, suppose $s \in \mathcal{K}_\vartheta \cup \mathcal{L}_\vartheta \cup \mathcal{S}_\vartheta$. We claim $s \not\succ \bar{b}$ and hence we can use eNfStrt:

$$\frac{s \in \mathcal{K}_\vartheta \cup \mathcal{L}_\vartheta \cup \mathcal{S}_\vartheta \quad s \not\succ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \quad (s_i \in \mathcal{N}_{\vartheta \cup \bar{x_i}})_{i \in I}}{\mathtt{case}\ s\ \mathtt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \in \mathcal{E}_\vartheta}\ \textsc{eNfStrt}$$

    We are left to prove the claim. If $s \in \mathcal{S}_\vartheta$, then it holds trivially since weak structures are not answers. If $s \in \mathcal{L}_\vartheta$, then trivially $s \not\succ \bar{b}$. If $s \in \mathcal{K}_\vartheta$, then $s \succ \bar{b}$ implies t not in normal form, contradicting our hypothesis.
  - For item 2, we consider two cases.
    * $x \in \mathsf{ngv}(s)$. Since $s \in \mathsf{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$ by the *i.h.* w.r.t. item 1, and $x \in \mathsf{ngv}(s)$, then by the *i.h.* w.r.t. item 2, $\exists \mathtt{C}_1, h_1$ s.t. $s = \mathtt{C}_1[\![x]\!]$, where $\mathtt{C}_1 \in \mathcal{C}^{h_1}_{\vartheta}$ and $x \notin \vartheta$. If $h_1 \notin \{\mathbf{c}_i\}_{i \in 1..n}$ then $\mathtt{case}\ \mathtt{C}_1\ \mathtt{of}\ \bar{b} \in \mathcal{C}^{\cdot}_{\vartheta}$ by eCase1, concluding the case. Suppose that $h_1 = \mathbf{c}_j \in \{\mathbf{c}_i\}_{i \in 1..n}$. Then by Lem. 45 for any term t, there exist $\mathtt{A}$ and $\mathtt{L}$ s.t. $\mathtt{C}_1[t] = \mathtt{A}[\mathbf{c}_j]\mathtt{L}$. In particular, $\mathtt{C}_1[x] = \mathtt{A}[\mathbf{c}_j]\mathtt{L}$. But then $|\mathtt{A}| = |\mathtt{A}(\mathtt{C}, x)| \neq |\bar{x_j}|$ for otherwise t would be a $\mathtt{case}$ redex, contradicting the hypothesis. Therefore, we then conclude by eCase1.

$$\frac{\mathtt{C} \in \mathcal{C}^{h}_{\vartheta} \quad h \notin \{\mathbf{c}_i\}_{i \in I} \text{ or } h = \mathbf{c}_j \in \{\mathbf{c}_i\}_{i \in I} \text{ and } |\mathtt{A}(\mathtt{C}, y)| \neq |\bar{x_j}|}{\mathtt{case}\ \mathtt{C}\ \mathtt{of}\ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \in \mathcal{C}^{\cdot}_{\vartheta}}\ \textsc{eCase1}$$

    * $x \in \mathsf{ngv}(s_i)$ with $i \in 1..n$. From the *i.h.* w.r.t. item 1, $s \in \mathcal{N}_\vartheta$ and $(s_i \in \mathcal{N}_{\vartheta \cup \{\bar{x_i}\}})_{i \in I}$. Therefore, by the *i.h.* w.r.t. item 2, $\exists \mathtt{C}_1, h_1$ s.t. $s_j = \mathtt{C}_1[\![x]\!]$, where $\mathtt{C}_1 \in \mathcal{C}^{h_1}_{\vartheta}$ and $x \notin \vartheta$. Moreover, $s \not\succ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I}$ must hold since we would have a $\mathtt{case}$ redex for the branch enabled by s otherwise. Therefore, we conclude by an application of eCase2.

$$\frac{s \in \mathcal{N}_\vartheta \quad s \not\succ (\mathbf{c}_i \bar{x}_i \Rightarrow s_i)_{i \in I} \quad s_k \in \mathcal{N}_{\vartheta \cup \bar{x_k}} \text{ for all } k < j \quad \mathtt{C} \in \mathcal{C}^{h}_{\vartheta \cup \bar{x_i}}}{\mathtt{case}\ s\ \mathtt{of}\ \mathbf{c}_1 \bar{x_1} \Rightarrow s_1, \ldots, \mathbf{c}_j \bar{x_j} \Rightarrow \mathtt{C}, \ldots, \mathbf{c}_n \bar{x_n} \Rightarrow s_n \in \mathcal{C}^{\cdot}_{\vartheta}}\ \textsc{eCase2}$$

- $t = t_1[y\backslash t_2]$.
  - For item 1, let ϑ be any set of variables such that $\mathsf{fv}(t_1[y\backslash t_2]) \subseteq \vartheta$. By the *i.h.*, we have $t_1 \in \mathcal{N}_{\vartheta\cup\{y\}}$ since $\mathsf{fv}(t_1) \subseteq \vartheta \cup \{y\}$ and $t_2 \in \mathcal{N}_\vartheta$ since $\mathsf{fv}(t_2) \subseteq \vartheta$.
    If $x \notin \mathsf{ngv}(t_1)$, then from Lem. 62, $t_1 \in \mathcal{N}_\vartheta$. We conclude $t_1[y\backslash t_2] \in \mathcal{N}_\vartheta$ using NFSUBG.

    $$\frac{t_1 \in \mathbb{X}_\vartheta \quad y \notin \mathsf{ngv}(t_1)}{t_1[y\backslash t_2] \in \mathbb{X}_\vartheta}\ \textsc{nfSubG}$$

    Suppose, on the contrary, that $y \in \mathsf{ngv}(t_1)$. From $t_2 \in \mathcal{N}_\vartheta$, it must be the case that $t_2 \in \mathcal{L}_\vartheta \cup \mathcal{S}_\vartheta \cup \mathcal{K}_\vartheta \cup \mathcal{E}_\vartheta$. From item 2 applied to t_1, we know $t_1 = \mathtt{C}[\![y]\!]$, where $\mathtt{C} \in \mathcal{C}^h_\vartheta$ and $y \notin \vartheta$. Then $t_2 \in \mathcal{L}_\vartheta \cup \mathcal{K}_\vartheta$ is not possible since t is in normal form. Hence $t_2 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$. We thus conclude by using:

    $$\frac{t_1 \in \mathbb{X}_{\vartheta\cup\{y\}} \quad t_2 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta \quad y \in \mathsf{ngv}(t_1)}{t_1[y\backslash t_2] \in \mathbb{X}_\vartheta}\ \textsc{nfSub}$$

  - For item 2 we consider two cases.
    * $x \in \mathsf{ngv}(t_1)$. Since $t_1 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$ by the *i.h.* w.r.t. item 1 and $x \in \mathsf{ngv}(t_1)$, by the *i.h.* w.r.t. item 2, $\exists \mathtt{C}_1, h_1$ s.t. $t_1 = \mathtt{C}_1[\![x]\!]$, where $\mathtt{C}_1 \in \mathcal{C}^{h_1}_\vartheta$ and $x \notin \vartheta$. Therefore, if $t_2 \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ we conclude by applying ESUBSLNONSTRUCT.

      $$\frac{\mathtt{C}_1 \in \mathcal{C}^h_\vartheta \quad t_2 \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta \quad y \notin \vartheta}{\mathtt{C}_1[y\backslash t_2] \in \mathcal{C}^h_\vartheta}\ \textsc{eSubsLNonStruct}$$

      Otherwise, if $t_2 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$, we conclude by first applying Lem. 64 and then use ESUBSLSTRUCT.

      $$\frac{\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{y\}} \quad t_2 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta}{\mathtt{C}_1[y\backslash t_2] \in \mathcal{C}^h_\vartheta}\ \textsc{eSubsLStruct}$$

    * $x \in \mathsf{ngv}(t_2)$. From the definition of $\mathsf{ngv}()$, it must be the case that $y \in \mathsf{ngv}(t_1)$. Since $t_2 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$ by the *i.h.* w.r.t. item 1 and $x \in \mathsf{ngv}(t_2)$, by the *i.h.* w.r.t. item 2, $\exists \mathtt{C}_2, h_2$ s.t. $t_2 = \mathtt{C}_2[\![x]\!]$, where $\mathtt{C}_2 \in \mathcal{C}^{h_2}_\vartheta$ and $x \notin \vartheta$. Additionally, since $t_1 \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$ by the *i.h.* w.r.t. item 1 and $y \in \mathsf{ngv}(t_1)$, by the *i.h.* w.r.t. item 2, $\exists \mathtt{C}_1, h_1$ s.t. $t_1 = \mathtt{C}_1[\![y]\!]$, where $\mathtt{C}_1 \in \mathcal{C}^{h_1}_\vartheta$ and $y \notin \vartheta$. Furthermore, by Lem. 45 h_2 must be '$\cdot$'. If $h = \mathbf{c}$, then, for any term t, there exist $\mathtt{A}$ and $\mathtt{L}$ s.t. $\mathtt{C}[t] = \mathtt{A}[\mathbf{c}]\mathtt{L}$. If $h = \lambda$, then, for any term t, there exists a variable x, term s and substitution context $\mathtt{L}$ s.t. $\mathtt{C}[t] = (\lambda x.s)\mathtt{L}$. In either of these cases we would have a redex, contradicting our hypothesis that $t \in \mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}})$. Therefore, we conclude by ESUBS-R.

      $$\frac{\mathtt{C}_1 \in \mathcal{C}^h_\vartheta \quad \mathtt{C}_2 \in \mathcal{C}^{\cdot}_\vartheta}{\mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}_2] \in \mathcal{C}^h_\vartheta}\ \textsc{eSubs-R}$$

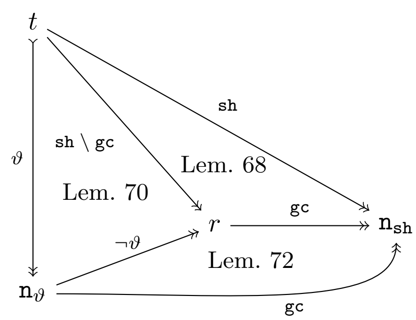
□

From Lem. 65 and Lem. 66 we deduce:

Lemma 67. $\mathsf{NF}(\rightarrowtail^\vartheta_{\mathtt{sh}}) = \mathcal{N}_\vartheta$.

6 A Standardization Theorem for the Theory of Sharing

This section addresses a standardization theorem for $\lambda_{\mathtt{sh}}$. Suppose t is definable in $\lambda_{\mathtt{sh}}$ as $\mathtt{n_{sh}} \in \mathcal{N}$. Then there is a reduction sequence $t \twoheadrightarrow_{\mathtt{sh}} \mathtt{n_{sh}}$ (*cf.* figure below). Notice that reduction steps in this sequence can take place under any context and substitution can take place even though the target is not needed for computing the strong normal-form. The standardization theorem reorganizes the computation steps in the reduction sequence $t \twoheadrightarrow_{\mathtt{sh}} \mathtt{n_{sh}}$ so that it can be factored into two parts $t \rightarrowtail\!\!\!\twoheadrightarrow^{\vartheta}_{\mathtt{sh}} u \twoheadrightarrow^{\neg\vartheta}_{\mathtt{sh}} \mathtt{n_{sh}}$. The prefix $t \rightarrowtail\!\!\!\twoheadrightarrow^{\vartheta}_{\mathtt{sh}} u$ is reduction via the strong-call-by-need strategy; the double headed arrow indicates multiple steps of the strategy. The suffix $u \twoheadrightarrow^{\neg\vartheta}_{\mathtt{sh}} \mathtt{n_{sh}}$ consists of reduction steps in the theory that are *internal*, hence not required for obtaining the strong-normal form. In fact, $\mathtt{n_{sh}}$ and u are shown to be identical via unsharing. Moreover, u is actually a normal form of the strategy. In summary, and following [BBBK17], the standardization theorem is split into three parts depicted below:



- Part I (Postponing $\mathtt{gc}$): All $\mathtt{gc}$ steps are postponed (Lem. 68).
- Part II (Factorization): The resulting prefix is factorized into an external part that contributes to the strong normal form and an internal part that does not (Lem. 70).
- Part III (Internal steps are negligible). Internal steps all take place inside garbage explicit substitutions (Lem. 72).

Part I Part I is just Lem. 68 below. A **strict $\rightarrow_{\mathtt{sh}}$-reduction step**, denoted $\rightarrow_{\mathtt{sh}\backslash\mathtt{gc}}$, is a $\rightarrow_{\mathtt{sh}}$-reduction step without using the $\mapsto_{\mathtt{gc}}$-rule.

Lemma 68 (Postponement of $\mathtt{gc}$)**.** *If $t \twoheadrightarrow_{\mathtt{sh}} s$, then there is a term u s.t. $t \twoheadrightarrow_{\mathtt{sh}\backslash\mathtt{gc}} u \twoheadrightarrow_{\mathtt{gc}} s$.*

Proof. Straightforward by noticing that the non $\mathtt{gc}$ reduction rules of $\lambda_{\mathtt{sh}}$ (Def. 10) allow the patterns of the left hand-sides to be polluted with explicit substitutions and do not place conditions on free variables. □

Part II Part II requires that we first define what an internal step is. **A ϑ-internal $\rightarrow_{\mathtt{sh}}$ step** ($\rightarrow^{\neg\vartheta}_{\mathtt{sh}}$) is a $\rightarrow_{\mathtt{sh}}$-step that is not a $\rightarrowtail^{\vartheta}_{\mathtt{sh}}$, *i.e.* not a ϑ-step in the strategy). **External steps** are steps in the strategy, that is, $\rightarrowtail^{\vartheta}_{\mathtt{sh}}$-steps.

Lemma 69 (Commutation of Internal/External Steps)**.** *Let* $\mathsf{fv}(t_0) \subseteq \vartheta$*. If* $t_0 \rightarrow^{\neg\vartheta}_{\mathtt{sh}} t_1 \rightarrowtail^{\vartheta}_{\mathtt{sh}} t_3$*, then there is a term* t_2 *such that* $t_0 \rightarrowtail\!\!\!\!\twoheadrightarrow^{\vartheta}_{\mathtt{sh}} t_2 \twoheadrightarrow^{\neg\vartheta}_{\mathtt{sh}} t_3$*, where the reduction from* t_0 *to* t_2 *includes at least one step and the one from* t_2 *to* t_3 *has at most two steps.*

Proof. The proof of this result is tedious and extensive. It has been relegated to the appendix. $\square$

The final factorization result is obtained from the commutation lemma. More precisely, one shows that $(\rightarrow^{\neg\vartheta}_{\mathtt{dB,fix,case}}, \rightarrowtail^{\vartheta}_{\mathtt{dB,fix,case}}, \rightarrow^{\neg\vartheta}_{\mathtt{lsv}}, \rightarrowtail^{\vartheta}_{\mathtt{lsv}})$ forms a *square factorization system* according to the terminology of [Acc12], taking $\rightarrowtail^{\vartheta}_{\mathtt{dB,fix,case}}$ (resp.. $\rightarrowtail^{\vartheta}_{\mathtt{lsv}}$) to be external $\mathtt{dB}$, $\mathtt{fix}$ or $\mathtt{case}$ (resp. $\mathtt{lsv}$) reduction, and $\rightarrow^{\neg\vartheta}_{\mathtt{dB,fix,case}}$ (resp. $\rightarrow^{\neg\vartheta}_{\mathtt{lsv}}$) to be the internal $\mathtt{dB}$, $\mathtt{fix}$ or $\mathtt{case}$ (resp. $\mathtt{lsv}$) reduction. One then concludes from Theorem 5.2 in [Acc12].

Lemma 70 (Factorization of Strict Steps)**.** *Let* $\mathsf{fv}(t) \subseteq \vartheta$*. If* $t \twoheadrightarrow_{\mathtt{sh}\setminus\mathtt{gc}} u$*, then there is a term* s *such that* $t \rightarrowtail\!\!\!\!\twoheadrightarrow^{\vartheta}_{\mathtt{sh}} u \twoheadrightarrow^{\neg\vartheta}_{\mathtt{sh}} s$*.*

Part III As mentioned, if the $\rightarrow_{\mathtt{sh}}$ reduction sequence reaches a $\rightarrow_{\mathtt{sh}}$-normal form, then all the internal steps factored out by Lem. 70 can be erased by $\mathtt{gc}$ steps.

Lemma 71 (Inclusion of Normal Forms)**.** *Let* ϑ*,* t *be such that* $\mathsf{fv}(t) \subseteq \vartheta$*. If* $t \in \mathtt{NF}(\rightarrow_{\mathtt{sh}})$*, then* $t \in \mathtt{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$*.*

Proof. This is immediate since $\rightarrowtail^{\vartheta}_{\mathtt{sh}} \subseteq \rightarrow_{\mathtt{sh}}$. $\square$

Lemma 72 (Normal Forms Modulo Internal and $\mathtt{gc}$ steps)**.** *Let* ϑ*,* t *be such that* $\mathsf{fv}(t) \subseteq \vartheta$*.*

1. *If* $t \rightarrow_{\mathtt{gc}} \mathtt{n}_{\vartheta}$ *with* $\mathtt{n}_{\vartheta} \in \mathtt{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$ *then* $t \in \mathtt{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$*.*
2. *If* $t \rightarrow^{\neg\vartheta}_{\mathtt{sh}} \mathtt{n}_{\vartheta}$ *with* $\mathtt{n}_{\vartheta} \in \mathtt{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$ *then* $t \in \mathtt{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$ *and there is* u *such that* $t \twoheadrightarrow_{\mathtt{gc}} u$ *and* $\mathtt{n}_{\vartheta} \twoheadrightarrow_{\mathtt{gc}} u$*.*

Proof. We show that the following conditions are equivalent for any term t such that $\mathsf{fv}(t) \subseteq \vartheta$. They imply items (1) and (2) of this lemma: (i) t is a $\rightarrowtail^{\vartheta}_{\mathtt{sh}}$-normal form, (ii) $\downarrow_{\mathtt{gc}}(t)$, that is t with all garbage substitutions removed, is a $\rightarrow_{\mathtt{sh}}$-normal form, (iii) $t \equiv_{\neg\vartheta} s$ for some $s \in \mathtt{NF}(\rightarrow_{\mathtt{sh}})$, (iv) $t \equiv_{\neg\vartheta} s$ for some $s \in \mathtt{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$. Here $\equiv_{\neg\vartheta}$ stands for the least equivalence relation containing $\rightarrow_{\mathtt{gc}} \cup \rightarrow^{\neg\vartheta}_{\mathtt{sh}}$. $\square$

Note that this result applies to terms in $\rightarrow_{\mathtt{sh}}$-normal form too, since $\mathtt{NF}(\rightarrow_{\mathtt{sh}}) \subseteq \mathtt{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$. Here is an example [BBBK17]. Consider the sequence $x[y\backslash z[z\backslash \mathtt{id}]] \rightarrow^{\neg\vartheta}_{\mathtt{sh}} x[y\backslash \mathtt{id}[z\backslash \mathtt{id}]] \rightarrow_{\mathtt{gc}} x$. All three terms are in $\mathtt{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$: this is straightforward for x, and due to the fact that the substitution is garbage for the two others. Moreover, although we do not have $x[y\backslash z[z\backslash \mathtt{id}]] \rightarrow_{\mathtt{gc}} x[y\backslash \mathtt{id}[z\backslash \mathtt{id}]]$, both terms reduce in one $\mathtt{gc}$-step to the same term x.

All three parts can now be assembled to complete the argument outlined in the Introduction.

Theorem 73 (Standardization for $\rightarrow_{\mathtt{sh}}$)**.** *Let* ϑ*,* t *be such that* $\mathsf{fv}(t) \subseteq \vartheta$*. If* $t \twoheadrightarrow_{\mathtt{sh}} \mathtt{n}_{\mathtt{sh}}$*, where* $\mathtt{n}_{\mathtt{sh}} \in \mathtt{NF}(\rightarrow_{\mathtt{sh}})$*, then there exists a term* $\mathtt{n}_{\vartheta} \in \mathtt{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$ *such that* $t \rightarrowtail\!\!\!\!\twoheadrightarrow^{\vartheta}_{\mathtt{sh}} \mathtt{n}_{\vartheta}$ *and* $\mathtt{n}_{\vartheta} \twoheadrightarrow_{\mathtt{gc}} \mathtt{n}_{\mathtt{sh}}$*.*

Proof. Suppose $t \twoheadrightarrow_{\mathtt{sh}} \mathtt{n}_{\mathtt{sh}}$ with $\mathtt{n}_{\mathtt{sh}} \in \mathtt{NF}(\rightarrow_{\mathtt{sh}})$ (so that in particular $\mathtt{n}_{\mathtt{sh}} \in \mathtt{NF}(\rightarrow_{\mathtt{gc}})$). By Lem. 68 of **Part I** there is a term r such that the reduction sequence $t \twoheadrightarrow_{\mathtt{sh}} \mathtt{n}_{\mathtt{sh}}$ can be decomposed as $t \twoheadrightarrow_{\mathtt{sh}\setminus\mathtt{gc}} r \twoheadrightarrow_{\mathtt{gc}} \mathtt{n}_{\mathtt{sh}}$, and by Lem. 70 of **Part II** there is a term u such that the reduction sequence $t \twoheadrightarrow_{\mathtt{sh}\setminus\mathtt{gc}} r$ can in turn be decomposed as $t \rightarrowtail\!\!\!\!\twoheadrightarrow^{\vartheta}_{\mathtt{sh}} u \twoheadrightarrow^{\neg\vartheta}_{\mathtt{sh}} r$. Finally, since $\mathsf{fv}(\mathtt{n}_{\mathtt{sh}}) \subseteq \varphi$ and

$\mathtt{NF}(\to_{\mathtt{sh}}) \subseteq \mathtt{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$, we have $\mathtt{n}_{\mathtt{sh}} \in \mathsf{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$, and Lem. 72 of **Part III** allows us to deduce that both r and u are also in $\mathsf{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$. Moreover, using convergence of $\to_{\mathtt{gc}}$ (which is straightforward), Lem. 72 further allows us to deduce that $u \twoheadrightarrow_{\mathtt{gc}} \mathtt{n}_{\mathtt{sh}}$. □

Corollary 74 (Completeness of $\rightarrowtail^{\vartheta}_{\mathtt{sh}}$)**.** *Let ϑ, t be such that $\mathsf{fv}(t) \subseteq \vartheta$. If $t \twoheadrightarrow_{\mathtt{e}} \mathtt{n}_{\mathtt{e}}$, where $\mathtt{n}_{\mathtt{e}} \in \mathtt{NF}(\to_{\mathtt{e}})$, then there exists a term $\mathtt{n}_{\vartheta} \in \mathtt{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}})$ such that $t \twoheadrightarrow^{\vartheta}_{\mathtt{sh}} \mathtt{n}_{\vartheta}$ and $\mathtt{n}^{\diamond}_{\vartheta} = \mathtt{n}_{\mathtt{e}}$.*

7 Related Work and Conclusions

Related Work Call-by-need for weak reduction was introduced in the 70s [Wad71, HM76]. Relating call-by-need strategies with call-by-need theories has been pioneered in [AF97, MOW98, CF12]. Big-step semantics for call-by-need was studied in [Lau93]. Completeness of call-by-need through intersection types was first studied in [Kes16], although the result itself was proved by other means before that [AF97]. A recent survey on non-idempotent intersection types and its applications in the study of the lambda calculus may be found here [BKV17]. The calculi with explicit substitutions at a distance used here is called the *Linear Substitution Calculus* and was inspired from [Mil07] and further developed in [AK10]. The use of this tool to study abstract machines for weak call-by-need reduction appears here [ABM14]. It is also used in [AB17], to provide a detailed analysis of the cost of adding pattern matching to β-reduction, although open terms are not considered.

Regarding strong reduction, as already mentioned in the introduction, [GL02] proposed an implementation of strong call-by-value, by iterating the standard call-by-value strategy on open terms (terms with variables). In [BDG11], it is noted that the implementation of [GL02] requires modifying the OCaml abstract machine so they propose a native OCaml implementation where the tags that distinguish functions from accumulators are coded directly in OCaml itself. [Cré90, Cré07] defined abstract machines for reduction to strong normal form. Other abstract machines for strong reduction have been studied too: [GNM13, dC09, ER06]. [AG16] explore open call-by-value and [ABM15] study a (call-by-name) machine based on the linear substitution calculus for reduction to strong normal form. None of these mentioned works address however strong call-by-need except for [BBBK17]. The latter proves similar results to this work but for β-reduction only. While developing this work we learned of [Ber14]. In his PhD thesis, Bernadet proposes a non-idempotent intersection type system for a similar calculus that includes fixed-points and case expressions. The aim however is to characterize a subset of strongly normalising terms. Thus, for example, the standard fixed-point combinator used here cannot be typed; a modified combinator is adopted. Since there is no notion of call-by-need reduction strategy, ideas related to good or covered types, as presented here, are not developed either.

Conclusions The recent formulation of a strong call-by-need strategy [BBBK17] was argued to provide a foundation for checking conversion in proof assistants. This work emerged out of the realization that the restriction to β-reduction of [BBBK17], and hence lack of treatment of inductive types and fixed point operators, left a gap to be filled. We have introduced a strong call-by-need strategy that is proved to be complete with respect to the Extended Lambda Calculus of Grégoire and Leroy [GL02] that includes the aforementioned constructs. A key obstacle has been devising a non-idempotent intersection type system that could connect reduction in the Extended Lambda Calculus with reduction in the theory of sharing, the latter is also introduced in this paper. This system is able to deal with case expressions that can block on open terms or non-exhaustive branches and also that can collect arguments. The presence of the fixed-point combinator has not provided any substantial obstacles.

In order to base an implementation of conversion in a proof assistant on our strategy, one should be able to iterate a restriction of it, to weak head normal form, as described in [Coq96]. This has the benefit of failing early when types are not equivalent. Another line of work is to implement a compiled version of the strategy, as developed in [GL02]. Finally, big-step semantics and abstract machines that implement our strategy are yet to be developed.

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A Postponement of internal steps

In this section we use the notion of the **anchor** of a step, which is the underlined variable in each of the following cases:

1. **$\mathtt{dB}$ step.**
$$\mathtt{C}[(\lambda \underline{x}.t)\mathtt{L}\, s]$$
i.e., the anchor is the binding occurrence of the variable bound by the abstraction that takes part in the pattern of the $\mathtt{dB}$-redex.

2. **$\mathtt{lsv}$ step.**
$$\mathtt{C}_1[\mathtt{C}_2[\![\underline{x}]\!][x \backslash v\mathtt{L}]]$$
i.e., the anchor is the occurrence of the variable that is substituted by the $\mathtt{lsv}$ step.

3. **$\mathtt{fix}$ step.**
$$\mathtt{C}[(\mathtt{fix}(\underline{x}.t))]$$
i.e., the anchor is the binding occurrence of the variable bound by fix that takes part in the pattern of the $\mathtt{fix}$-redex.

4. **$\mathtt{case}$ step.**
$$\mathtt{C}[\mathtt{case}\ \underline{\mathtt{A}[\mathbf{c}_j]\mathtt{L}}\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}]$$
i.e., the anchor is target of the case that takes part in the pattern of the $\mathtt{case}$-redex.

Definition 75 (Internal and external steps)**.** *We say that t_1 reduces in a ϑ-**internal step** to t_2, written $t_1 \rightarrow^{\neg\vartheta}_{\mathtt{sh}} t_2$, if and only if there is a step in call-by-need that is not a $\mathtt{gc}$ step and is not in the strong call-by-need strategy,* i.e. $t_1\ (\rightarrow_{\mathtt{sh}\backslash\mathtt{gc}} \backslash \rightarrowtail^{\vartheta}_{\mathtt{sh}})\ t_2$. *We sometimes call ϑ-internal steps just **internal steps** if ϑ is clear from the context. Steps in the strategy $\rightarrowtail^{\vartheta}_{\mathtt{sh}}$ are called ϑ-**external steps** (or just **external steps**).*

Lemma 76 (Substitution contexts are evaluation contexts)**.** *If $\mathtt{L}$ is a substitution context s.t. $\operatorname{dom}\mathtt{L} \cap \vartheta = \varnothing$, then exists h s.t. $\mathtt{L} \in \mathcal{C}^h_\vartheta$.*

Proof. By induction on $\mathtt{L}$. The empty case is immediate from X-Box. If $\mathtt{L} = \mathtt{L}'[x\backslash t]$ consider two cases, depending on whether t is a weak structure or weak error term or not:

1. **If $t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** By *i.h.* we have that $\mathtt{L}' \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ so $\mathtt{L}'[x\backslash t] \in \mathcal{C}^h_\vartheta$.
2. **If $t \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** By *i.h.* we have that $\mathtt{L}' \in \mathcal{C}^h_\vartheta$. By the hypothesis we can assume $x \notin \vartheta$. Hence $\mathtt{L}'[x\backslash t] \in \mathcal{C}^h_\vartheta$ follows from ESubsLNonStr.

□

Lemma 77 (Answers are stable by reduction)**.** *1. Let $(\lambda x.t)\mathtt{L} \to_{\mathtt{sh}\backslash\mathtt{gc}} s$ be a* $\mathtt{dB}$*,* $\mathtt{lsv}$*,* $\mathtt{fix}$*, or* $\mathtt{case}$ *step. Then s is an abstraction answer.*

2. Let $\mathtt{A}[\mathbf{c}]\mathtt{L} \to_{\mathtt{sh}\backslash\mathtt{gc}} s$ be a $\mathtt{dB}$*,* $\mathtt{lsv}$*,* $\mathtt{fix}$*, or* $\mathtt{case}$ *step. Then s is a constructor answer.*

Proof. Both items are proved by cases on the kind of step contracted. We present a detailed proof of first case and then comment on the second one (whose proof is similar).

1. $\mathtt{dB}$ **step.** Let Δ denote the $\mathtt{dB}$-redex and Δ' its contractum. Two cases: the step is either internal to t or internal to one of the substitutions in $\mathtt{L}$:
   (a) **If the step is internal to t.** Then $t = \mathtt{C}[\Delta]$ and the step is of the form:
   $$(\lambda x.\mathtt{C}[\Delta])\mathtt{L} \to_{\mathtt{sh}\backslash\mathtt{gc}} (\lambda x.\mathtt{C}[\Delta'])\mathtt{L}$$
   (b) **If the step is internal to one of the substitutions in $\mathtt{L}$.** Then $\mathtt{L} = \mathtt{L}_1[y\backslash\mathtt{C}[\Delta]]\mathtt{L}_2$ and the step is of the form:
   $$(\lambda x.t)\mathtt{L}_1[y\backslash\mathtt{C}[\Delta]]\mathtt{L}_2 \to_{\mathtt{sh}\backslash\mathtt{gc}} (\lambda x.t)\mathtt{L}_1[y\backslash\mathtt{C}[\Delta']]\mathtt{L}_2$$
2. $\mathtt{lsv}$ **step.** Two cases: the variable contracted by the $\mathtt{lsv}$ step is either internal to t or internal to one of the substitutions in $\mathtt{L}$.
   (a) **If the variable is internal to t.** Then $t = \mathtt{C}[y]$. Two subcases: the substitution that binds y is either in $\mathtt{C}$ or one of the substitutions in $\mathtt{L}$.
      i. **If the contracted substitution is in $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_1[\mathtt{C}_2[y\backslash v\mathtt{L}']]$ and the step is of the form:
      $$(\lambda x.\mathtt{C}_1[\mathtt{C}_2[\![y]\!][y\backslash v\mathtt{L}']])\mathtt{L} \to_{\mathtt{sh}\backslash\mathtt{gc}} (\lambda x.\mathtt{C}_1[\mathtt{C}_2[v][y\backslash v]\mathtt{L}'])\mathtt{L}$$
      ii. **If the contracted substitution is one of the substitutions in $\mathtt{L}$.** Then $\mathtt{L} = \mathtt{L}_1[y\backslash v\mathtt{L}']\mathtt{L}_2$ and the step is of the form:
      $$(\lambda x.\mathtt{C}[\![y]\!])\mathtt{L}_1[y\backslash v\mathtt{L}']\mathtt{L}_2 \to_{\mathtt{sh}\backslash\mathtt{gc}} (\lambda x.\mathtt{C}[v])\mathtt{L}_1[y\backslash v]\mathtt{L}'\mathtt{L}_2$$
   (b) **If the variable is internal to one of the substitutions in $\mathtt{L}$.** Then $\mathtt{L} = \mathtt{L}_1[z\backslash\mathtt{C}[y]]\mathtt{L}_2$. Two subcases: the substitution that binds y is either in $\mathtt{C}$ or one of the substitutions in $\mathtt{L}_2$.
      i. **If the contracted substitution is in $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_1[\mathtt{C}_2[y\backslash v\mathtt{L}']]$ and the step is of the form:
      $$(\lambda x.t)\mathtt{L}_1[z\backslash\mathtt{C}_1[\mathtt{C}_2[\![y]\!][y\backslash v\mathtt{L}']]]\mathtt{L}_2 \to_{\mathtt{sh}\backslash\mathtt{gc}} (\lambda x.t)\mathtt{L}_1[z\backslash\mathtt{C}_1[\mathtt{C}_2[v][y\backslash v]\mathtt{L}']]\mathtt{L}_2$$

ii. **If the contracted substitution is one of the substitutions in $\mathtt{L}_2$.** Then $\mathtt{L}_2 = \mathtt{L}_{21}[y\backslash v\mathtt{L}']\mathtt{L}_{22}$ and the step is of the form:

$$(\lambda x.t)\mathtt{L}_1[z\backslash \mathtt{C}[\![y]\!]]\mathtt{L}_{21}[y\backslash v\mathtt{L}']\mathtt{L}_{22} \to_{\mathtt{sh}\backslash\mathtt{gc}} (\lambda x.t)\mathtt{L}_1[z\backslash \mathtt{C}[v]]\mathtt{L}_{21}[y\backslash v]\mathtt{L}'\mathtt{L}_{22}$$

3. **`fix` step.** Let Δ denote the `fix`-redex and Δ' its contractum. Two cases: the step is either internal to t or internal to one of the substitutions in $\mathtt{L}$:

   (a) **If the step is internal to t.** Then $t = \mathtt{C}[\Delta]$ and the step is of the form:

   $$(\lambda x.\mathtt{C}[\Delta])\mathtt{L} \to_{\mathtt{sh}\backslash\mathtt{gc}} (\lambda x.\mathtt{C}[\Delta'])\mathtt{L}$$

   (b) **If the step is internal to one of the substitutions in $\mathtt{L}$.** Then $\mathtt{L} = \mathtt{L}_1[y\backslash \mathtt{C}[\Delta]]\mathtt{L}_2$ and the step is of the form:

   $$(\lambda x.t)\mathtt{L}_1[y\backslash \mathtt{C}[\Delta]]\mathtt{L}_2 \to_{\mathtt{sh}\backslash\mathtt{gc}} (\lambda x.t)\mathtt{L}_1[y\backslash \mathtt{C}[\Delta']]\mathtt{L}_2$$

4. **`case` step.** Let Δ denote the `case`-redex and Δ' its contractum. Two cases: the step is either internal to t or internal to one of the substitutions in $\mathtt{L}$:

   (a) **If the step is internal to t.** Then $t = \mathtt{C}[\Delta]$ and the step is of the form:

   $$(\lambda x.\mathtt{C}[\Delta])\mathtt{L} \to_{\mathtt{sh}\backslash\mathtt{gc}} (\lambda x.\mathtt{C}[\Delta'])\mathtt{L}$$

   (b) **If the step is internal to one of the substitutions in $\mathtt{L}$.** Then $\mathtt{L} = \mathtt{L}_1[y\backslash \mathtt{C}[\Delta]]\mathtt{L}_2$ and the step is of the form:

   $$(\lambda x.t)\mathtt{L}_1[y\backslash \mathtt{C}[\Delta]]\mathtt{L}_2 \to_{\mathtt{sh}\backslash\mathtt{gc}} (\lambda x.t)\mathtt{L}_1[y\backslash \mathtt{C}[\Delta']]\mathtt{L}_2$$

For the second case note that a term $\mathtt{A}[\mathbf{c}]\mathtt{L}$ is of the form $..((((\mathbf{c}\mathtt{L}_1\, t_1)\mathtt{L}_2)\, t_3)\mathtt{L}_3 \ldots t_n)\mathtt{L}_n$. Thus any step which takes place either in the t_i or in the $\mathtt{L}_i$ preserves this same form. In the case of a `lsv`-step that substitutes into a t_i or $\mathtt{L}_i$, again the same form is preserved. □

Lemma 78 (Abstraction answers are backwards stable by internal steps)**.** *Let $t_0 \to_{\mathtt{sh}}^{\neg\vartheta} (\lambda x.s)\mathtt{L} = t$ be a ϑ-internal step. Then:*

- *the source of the step is of the form $t_0 = (\lambda x.s_0)\mathtt{L}_0$;*
- *the anchor of the step is not below a substitution context,* i.e. *it is inside s_0 or inside one of the arguments of $\mathtt{L}_0$.*

Proof. By induction on the context $\mathtt{C}$ under which the step takes place:

1. **Empty, $\mathtt{C} = \Box$.** Note that the step cannot be a `dB`, `fix`, nor `case` step, as it would then be a ϑ-external step, since $\Box \in \mathcal{C}^h_\vartheta$.

   So it must be a `lsv` step, contracting the outermost substitution, that is, $t_0 = \mathtt{C}_1[\![y]\!][y\backslash v\mathtt{L}_2] \to_{\mathtt{sh}}^{\neg\vartheta} \mathtt{C}_1[v][y\backslash v]\mathtt{L}_2 = t$. Note that $\mathtt{C}_1[v] = (\lambda x.s)\mathtt{L}_1$ where $\mathtt{L} = \mathtt{L}_1[y\backslash v]\mathtt{L}_2$.

   We claim that $\mathtt{C}_1$ is not a substitution context. By contradiction, suppose that $\mathtt{C}_1$ is a substitution context. Then the `lsv` step $t_0 = y\mathtt{L}'[y\backslash v\mathtt{L}_2] \to_{\mathtt{sh}}^{\neg\vartheta} v\mathtt{L}'[y\backslash v]\mathtt{L}_2 = t$ is ϑ-external since $\mathtt{L}'[y\backslash v\mathtt{L}_2] \in \mathcal{C}^h_\vartheta$ by Lem. 76. This contradicts the assumption that the step is ϑ-internal.

   Now, since $\mathtt{C}_1[v] = (\lambda x.s)\mathtt{L}_1$, there are two cases, depending on the position of the hole of $\mathtt{C}_1$:

(a) **The hole of $\mathtt{C}_1$ lies inside s.** Then $\mathtt{C}_1 = (\lambda x.\mathtt{C}_2)\mathtt{L}_1$, and the step is of the form $t_0 = (\lambda x.\mathtt{C}_2[\![y]\!])\mathtt{L}_1[y\backslash v\mathtt{L}_2] \to_{\mathtt{sh}}^{\neg\vartheta} (\lambda x.\mathtt{C}_2[v])\mathtt{L}_1[y\backslash v]\mathtt{L}_2 = t$. By taking $s_0 := \mathtt{C}_2[\![y]\!]$ and $\mathtt{L}_0 := \mathtt{L}_1[y\backslash v\mathtt{L}_2]$ we conclude.

(b) **The hole of $\mathtt{C}_1$ lies inside $\mathtt{L}_1$.** Then $\mathtt{C}_1 = (\lambda x.s)\mathtt{L}_{11}[z\backslash\mathtt{C}_2]\mathtt{L}_{12}$ where $\mathtt{L}_1 = \mathtt{L}_{11}[z\backslash\mathtt{C}_2[v]]\mathtt{L}_{12}$, and the step is of the form $t_0 = (\lambda x.s)\mathtt{L}_{11}[z\backslash\mathtt{C}_2[\![y]\!]]\mathtt{L}_{12}[y\backslash v\mathtt{L}_2] \to_{\mathtt{sh}}^{\neg\vartheta} (\lambda x.s)\mathtt{L}_{11}[z\backslash\mathtt{C}_2[v]]\mathtt{L}_{12}[y\backslash v]\mathtt{L}_2 = t$. By taking $s_0 := s$ and $\mathtt{L}_0 := \mathtt{L}_{11}[z\backslash\mathtt{C}_2[\![y]\!]]\mathtt{L}_{12}[y\backslash v\mathtt{L}_2]$ we conclude.

Note that, as already argued, in both cases, the anchor of the step is not below a substitution context.

2. **Inside an abstraction, $\mathtt{C} = \lambda x.\mathtt{C}'$.** The step is of the form $t_0 = \lambda x.\mathtt{C}'[r_0] \to_{\mathtt{sh}}^{\neg\vartheta} \lambda x.\mathtt{C}'[r] = t$, so $\mathtt{L} = \mathtt{L}_0 = \Box$, with $s_0 = \mathtt{C}'[r_0]$ and $s = \mathtt{C}'[r]$. Note that the anchor of the step is inside s_0, hence not below a substitution context.

3. **Left of an application, $\mathtt{C} = \mathtt{C}'\,u$.** Impossible, since the step would be of the form $t_0 = \mathtt{C}'[r_0]\,u \to_{\mathtt{sh}}^{\neg\vartheta} \mathtt{C}'[r]\,u = t$ but t is not an application.

4. **Right of an application, $\mathtt{C} = u\,\mathtt{C}'$.** Impossible, analogous to the previous case.

5. **Inside a fix, $\mathtt{C} = \mathtt{fix}(x.\mathtt{C}')$.** Impossible, analogous to the previous case.

6. **Inside the body of a case, $\mathtt{C} = \mathtt{case}\ \mathtt{C}'\ \mathtt{of}\ \bar{b}$.** Impossible, analogous to the previous case.

7. **Inside the branch of a case, $\mathtt{C} = \mathtt{case}\ t\ \mathtt{of}\ (\mathbf{c}_1\bar{x}_1 \Rightarrow s_1)\ldots(\mathbf{c}_i\bar{x}_i \Rightarrow \mathtt{C}')\ldots(\mathbf{c}_n\bar{x}_n \Rightarrow s_n)$.** Impossible, analogous to the previous case.

8. **Left of a substitution, $\mathtt{C} = \mathtt{C}'[y\backslash u]$.** Then the step is of the form $t_0 = \mathtt{C}'[r][y\backslash u] \to_{\mathtt{sh}}^{\neg\vartheta} \mathtt{C}'[r'][y\backslash u] = (\lambda x.s)\mathtt{L}'[y\backslash u] = t$, where $\mathtt{L} = \mathtt{L}'[y\backslash u]$. We consider two cases, depending on whether u is a strong ϑ-structure:

   (a) **If $u \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** Note that the isomorphic step $\mathtt{C}'[r] \to_{\mathtt{sh\backslash gc}} (\lambda x.s)\mathtt{L}'$, taking place under the context $\mathtt{C}'$, cannot be $(\vartheta \cup \{y\})$-external, since then the fact that $\mathtt{C}' \in \mathcal{C}^h_{\vartheta\cup\{y\}}$ would imply that $\mathtt{C}'[y\backslash u] \in \mathcal{C}^h_\vartheta$, and the original step would be ϑ-external, contradicting the hypothesis.
   Hence the step $\mathtt{C}'[r] \to_{\mathtt{sh}}^{\neg\vartheta\cup\{y\}} (\lambda x.s)\mathtt{L}'$ is $(\vartheta\cup\{y\})$-internal. By *i.h.* we have that $\mathtt{C}'[r] = (\lambda x.s_0)\mathtt{L}'_0$, so the source of the original step is of the form $\mathtt{C}'[r][y\backslash u] = (\lambda x.s_0)\mathtt{L}'_0[y\backslash u]$. By *i.h.*, we also have that the anchor of the step is either inside s_0, or inside one of the arguments of $\mathtt{L}'_0$ By taking $\mathtt{L}_0 := \mathtt{L}'_0[y\backslash u]$ we conclude.

   (b) **If $u \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** Similar to the previous case: the isomorphic step $\mathtt{C}'[r] \to_{\mathtt{sh\backslash gc}} (\lambda x.s)\mathtt{L}'$, taking place under the context $\mathtt{C}'$, cannot be ϑ-external, as this would imply that the original step is ϑ-external.
   So it must be ϑ-internal and we may apply the *i.h.* to conclude that $\mathtt{C}'[r] = (\lambda x.s_0)\mathtt{L}'_0$ and, moreover, that the anchor of the step is either inside s_0, or inside one of the arguments of $\mathtt{L}'_0$. This means that the source of the original step is of the form $\mathtt{C}'[r][y\backslash u] = (\lambda x.s_0)\mathtt{L}'_0[y\backslash u]$, as required.

9. **Inside a substitution, $\mathtt{C} = u[y\backslash\mathtt{C}']$.** Then it must be the case that $u = (\lambda x.s)\mathtt{L}'$ and the step is of the form $(\lambda x.s)\mathtt{L}'[y\backslash\mathtt{C}'[r]] \to_{\mathtt{sh}}^{\neg\vartheta} (\lambda x.s)\mathtt{L}'[y\backslash\mathtt{C}'[r']]$, with $\mathtt{L} = \mathtt{L}'[y\backslash\mathtt{C}'[r']]$. By taking $s_0 := s$ and $\mathtt{L}_0 := \mathtt{L}'[y\backslash\mathtt{C}'[r]]$ we conclude. Note that the anchor of the step is inside one of the arguments of $\mathtt{L}_0$, as required.

$\square$

Lemma 79 (Beta-redexes are backwards stable by internal steps)**.** *Let* $t_0 \to_{\mathtt{sh}}^{\neg\vartheta} (\lambda x.s)\mathtt{L}\, u = t$ *be a* ϑ*-internal step. Then:*

- *the source of the step is of the form* $t_0 = (\lambda x.s_0)\mathtt{L}_0\, u_0$*;*
- *the anchor of the step is not below a context of the form* $\mathtt{L}'\, u_0$*,* i.e. *it is inside* s_0*, inside one of the arguments of* $\mathtt{L}_0$*, or inside* u_0*.*

Proof. By case analysis on the context $\mathtt{C}$ under which the step takes place:

1. **Empty,** $\mathtt{C} = \square$**.** We claim that this case is impossible.

   The step cannot be a $\mathtt{dB}$, $\mathtt{fix}$, nor $\mathtt{case}$ step, since $\square \in \mathcal{C}^h_\vartheta$, so it would be a ϑ-external step, contradicting the hypothesis that it is a ϑ-internal step.

   So it must be a $\mathtt{lsv}$ step, *i.e.* of the form $t_0 = \mathtt{C}_1[\![y]\!][y\backslash v\mathtt{L}'] \to_{\mathtt{sh}}^{\neg\vartheta} \mathtt{C}_1[v][y\backslash v]\mathtt{L}' = t$. This is impossible, as we would have $\mathtt{C}_1[v][y\backslash v]\mathtt{L}' = t = (\lambda x.s)\mathtt{L}\, u$ but the root of $\mathtt{C}_1[v][y\backslash v]\mathtt{L}'$ is a substitution node, while the root of $(\lambda x.s)\mathtt{L}\, u$ is an application node, which is a contradiction.

2. **Left of an application,** $\mathtt{C} = \mathtt{C}'\, u$**.** The step is of the form $\mathtt{C}'[r]\, u \to_{\mathtt{sh}}^{\neg\vartheta} \mathtt{C}'[r']\, u$. The isomorphic step $\mathtt{C}'[r] \to_{\mathtt{sh}\backslash\mathtt{gc}} \mathtt{C}'[r']$ takes place under the context $\mathtt{C}'$.

   We consider two cases, depending on whether $\mathtt{C}'$ is a evaluation context over ϑ or not:

   (a) **If** $\mathtt{C}' \in \mathcal{C}^h_\vartheta$**.** Note that $h = \lambda$ since otherwise $\mathtt{C}'\, u \in \mathcal{C}^h_\vartheta$, and the original step is ϑ-external, contradicting the hypothesis.
   By Lem. 45, since $\mathtt{C}' \in \mathcal{C}^\lambda_\vartheta$, we know that $\mathtt{C}'$ has the form of an abstraction answer, that is, more precisely, there are two cases:

      i. **The context** $\mathtt{C}'$ **is of the form** $(\lambda x.\mathtt{C}'')\mathtt{L}$**.** Then the original step is

      $$t_0 = (\lambda x.\mathtt{C}''[r])\mathtt{L}\, s \to_{\mathtt{sh}}^{\neg\vartheta} (\lambda x.\mathtt{C}''[r'])\mathtt{L}\, s = t$$

      By taking $s_0 := \mathtt{C}''[r]$, with $\mathtt{L}_0 = \mathtt{L}$ and $u_0 = u$ we conclude. Note that the anchor is internal to s_0, as required.

      ii. **The context** $\mathtt{C}'$ **is of the form** $(\lambda x.s)\mathtt{L}_1[y\backslash\mathtt{C}'']\mathtt{L}_2$**.** Then the original step is

      $$t_0 = (\lambda x.s)\mathtt{L}_1[y\backslash\mathtt{C}''[r]]\mathtt{L}_2\, s \to_{\mathtt{sh}}^{\neg\vartheta} (\lambda x.s)\mathtt{L}_1[y\backslash\mathtt{C}''[r']]\mathtt{L}_2\, s = t$$

      By taking $\mathtt{L}_0 := \mathtt{L}_1[y\backslash\mathtt{C}''[r]]\mathtt{L}_2$, with $s_0 = s$ and $u_0 = u$ we conclude. Note that the anchor is internal to one of the arguments of $\mathtt{L}_0$, as required.

   (b) **If** $\mathtt{C}' \notin \mathcal{C}^h_\vartheta$**.** Then the step $\mathtt{C}'[r] \to_{\mathtt{sh}\backslash\mathtt{gc}} \mathtt{C}'[r'] = (\lambda x.s)\mathtt{L}$ is ϑ-internal, so by the fact that answers are backwards stable by internal steps (Lem. 78) we have that $\mathtt{C}'[r] = (\lambda x.s_0)\mathtt{L}_0$ and the anchor of the step is not below a substitution context. Hence $t = \mathtt{C}'[r]\, u = (\lambda x.s_0)\mathtt{L}_0\, u$ and the anchor of the original step is not below a context of the form $\mathtt{L}\, u$, as required.

3. **Right of an application,** $\mathtt{C} = (\lambda x.s)\mathtt{L}\,\mathtt{C}'$**.** The step is of the form $t_0 = (\lambda x.s)\mathtt{L}\,\mathtt{C}'[r_0] \to_{\mathtt{sh}}^{\neg\vartheta} (\lambda x.s)\mathtt{L}\,\mathtt{C}'[r] = t$ By taking $u_0 := \mathtt{C}'[r_0]$, with $s_0 = s$ and $\mathtt{L}_0 = \mathtt{L}$ we conclude. Note that the anchor is internal to u_0, as required.

4. **Constructor other than an application, *i.e.*** $\mathtt{C} = \lambda x.\mathtt{C}'$**, or** $\mathtt{C} = \mathtt{C}'[y\backslash r]$**, or** $\mathtt{C} = r[y\backslash \mathtt{C}']$**, or** $\mathtt{C} = \mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \dots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}', \dots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n$**, or** $\mathtt{C} = \mathtt{case}\ \mathtt{C}'\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$**, or** $\mathtt{C} = \mathtt{fix}(x.\mathtt{C}')$**.** Impossible, as this would imply that the root of t is not an application node, but we know that $t = (\lambda x.s)\mathtt{L}\,u$ is an application.

□

Lemma 80 (Adding an arbitrary substitution preserves evaluation contexts)**.** *If $\mathtt{C} \in \mathcal{C}^h_\vartheta$ and $x \notin \vartheta$, then $\mathtt{C}[x\backslash t] \in \mathcal{C}^h_\vartheta$.*

Proof. Two cases, depending on whether t is a weak structure/error term or not:

1. **If** $t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$**.** By the weakening lemma for evaluation contexts (Lem. 64), the fact that $\mathtt{C} \in \mathcal{C}^h_\vartheta$ implies that $\mathtt{C} \in \mathcal{C}^h_{\vartheta\cup\{x\}}$. By applying the context formation rule for structural substitutions (ESubLStr) we have $\mathtt{C}[x\backslash t] \in \mathcal{C}^h_\vartheta$, as wanted.

2. **If** $t \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$**.** Recall that $x \notin \vartheta$ by hypothesis. By applying the context formation rule for non-s(ESubLStr)tructural substitutions (ESubLNonStr) $\mathtt{C} \in \mathcal{C}^h_\vartheta$ implies $\mathtt{C}[x\backslash t] \in \mathcal{C}^h_\vartheta$, as wanted.

□

Lemma 81 (Reachable variables after substitution of a variable)**.** *Let x be a variable not bound by a context $\mathtt{C}$. Then for any term t:*

$$\mathsf{ngv}(\mathtt{C}[\![x]\!]) \subseteq \mathsf{ngv}(\mathtt{C}[t]) \cup \{x\}$$

Proof. By induction on $\mathtt{C}$. □

Lemma 82 (Reachable variables in normal forms are below evaluation contexts)**.** *If $t \in \mathcal{N}_{\vartheta\cup\{x\}}$ and $x \in \mathsf{ngv}(t)$ then t can be written as $t = \mathtt{C}[\![x]\!]$ with $\mathtt{C} \in \mathcal{C}^h_\vartheta$.*

Proof. Corollary of Lem. 65 and Lem. 66(2).

□

Definition 83 ($\mathcal{C}^h_\vartheta$-critical contexts)**.** *Let $\mathcal{C}^h_\vartheta$ be a set of terms depending on a set of variables ϑ. A context $\mathtt{C}$ is said to be $\mathcal{C}^h_\vartheta$-*critical *if the following conditions hold:*

1. *$\mathtt{C}[p] \in \mathcal{C}^h_\vartheta$ for some term p; and*

2. *$\mathtt{C}[\![x]\!] \notin \mathcal{C}^h_\vartheta$ for some variable $x \notin \vartheta$ that is not bound by $\mathtt{C}$.*

Lemma 84 (Normal-form-critical contexts are evaluation contexts)**.** *The following inclusions between sets hold:*

1. *The set of $\mathcal{N}_\vartheta$-critical contexts is included in $\mathcal{C}^h_\vartheta$.*

2. *The set of $\mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$-critical contexts is included in $\mathcal{C}^{\cdot}_\vartheta$.*

3. *The set of $\mathcal{L}_\vartheta$-critical contexts is included in $\mathcal{C}^\lambda_\vartheta$.*

Proof. We address the first two items, the third being similar. Let $\mathbb{X}_\vartheta$ denote the set $\mathcal{N}_\vartheta$ (resp. $\mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$), and let Y^ϑ denote the set $\mathcal{C}^h_\vartheta$ (resp. $\mathsf{NACtxt}_\vartheta$). Suppose that $\mathtt{C}$ is a $\mathcal{C}^h_\vartheta$-critical context, and let us show that $\mathtt{C} \in Y^\vartheta$. Since $\mathtt{C}$ is $\mathcal{C}^h_\vartheta$-critical, there is a term p and a variable x not bound by $\mathtt{C}$ such that $\mathtt{C}[p] \in \mathcal{C}^h_\vartheta$ and $\mathtt{C}[\![x]\!] \notin \mathcal{C}^h_\vartheta$. We proceed by induction on the derivation that $\mathtt{C}[p] \in \mathcal{C}^h_\vartheta$:

1. **n-var,** $\mathtt{C}[p] = y \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$**.** Then $\mathtt{C} = \Box$ and in fact $\mathtt{C} \in \mathsf{NACtxt}_\vartheta$.
2. **nfLam,** $\mathtt{C}[p] = \lambda y.N^{\vartheta\cup\{y\}} \in \mathcal{N}_\vartheta$**.** If $\mathtt{C}$ is empty, *i.e.* $\mathtt{C} = \Box$, we trivially have $\mathtt{C} \in \mathcal{C}^h_\vartheta$. Otherwise, $\mathtt{C}$ is non-empty, *i.e.* $\mathtt{C} = \lambda y.\mathtt{C}'$ such that $\mathtt{C}'[p] \in \mathcal{N}_{\vartheta\cup\{y\}}$ and $\mathtt{C}'[\![x]\!] \notin \mathcal{N}_{\vartheta\cup\{y\}}$. By *i.h.* we conclude $\mathtt{C}' \in \mathcal{C}^h_{\vartheta\cup\{y\}}$, and hence $\lambda y.\mathtt{C}' \in \mathcal{C}^h_\vartheta$.
3. **n-app,** $\mathtt{C}[p] = M^\vartheta\, N^\vartheta \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$**.** If $\mathtt{C}$ is empty, *i.e.* $\mathtt{C} = \Box$, we trivially have $\mathtt{C} \in \mathsf{NACtxt}_\vartheta$. Otherwise, $\mathtt{C}$ is non-empty and there are two possibilities:
   (a) **The hole of $\mathtt{C}$ is to the left, *i.e.* $\mathtt{C} = \mathtt{C}'\, N^\vartheta$.** Then $\mathtt{C}'[p] \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ and $\mathtt{C}'[\![x]\!] \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$. By *i.h.* we obtain that $\mathtt{C}' \in \mathsf{NACtxt}_\vartheta$, so by applying the formation rule for generalized non-answer evaluation contexts over ϑ, using the left-application rule, we have $\mathtt{C}'\, N^\vartheta \in \mathcal{C}^{\cdot}_\vartheta$.
   (b) **The hole of $\mathtt{C}$ is to the right, *i.e.* $\mathtt{C} = M^\vartheta\, \mathtt{C}'$.** Then $\mathtt{C}'[p] \in \mathcal{N}_\vartheta$ and $\mathtt{C}'[\![x]\!] \notin \mathcal{N}_\vartheta$. By *i.h.* we obtain that $\mathtt{C}' \in \mathcal{C}^h_\vartheta$, so by applying the formation rule for generalized non-answer evaluation contexts over ϑ, using the right-application rule, we have $M^\vartheta\, \mathtt{C}' \in \mathcal{C}^{\cdot}_\vartheta$. Note that we must use the fact that M^ϑ is a strong ϑ-structure to be able to apply the formation rule.
4. **nfSubG,** $\mathtt{C}[p] = t[y\backslash s] \in \mathbb{X}_\vartheta$ **with** $t \in \mathcal{C}^h_\vartheta$ **and** $y \notin \mathsf{ngv}(t)$**.** Let us check that $\mathtt{C} \in Y^\vartheta$. If $\mathtt{C}$ is empty, *i.e.* $\mathtt{C} = \Box$, we trivially have $\mathtt{C} \in Y^\vartheta$. Otherwise, $\mathtt{C}$ is non-empty and there are two possibilities:
   (a) **The hole of $\mathtt{C}$ is to the left, *i.e.* $\mathtt{C} = \mathtt{C}'[y\backslash s]$.** Then $\mathtt{C}'[p] \in \mathcal{C}^h_\vartheta$ by formation of $\mathtt{C}[p] \in \mathcal{C}^h_\vartheta$. Moreover we claim that $\mathtt{C}'[\![x]\!] \notin \mathcal{C}^h_\vartheta$. To see this, note that the fact that $y \notin \mathsf{ngv}(\mathtt{C}'[p]) \cup \{x\}$ implies that $y \notin \mathsf{ngv}(\mathtt{C}'[\![x]\!])$ by the contrapositive of Lem. 81. So, by contradiction, if we suppose $\mathtt{C}'[\![x]\!] \notin \mathcal{C}^h_\vartheta$ we can apply the same formation rule and obtain that $\mathtt{C}'[\![x]\!][y\backslash s] \in \mathcal{C}^h_\vartheta$, contradicting the hypothesis that $\mathtt{C}[\![x]\!] \notin \mathcal{C}^h_\vartheta$.
   Therefore we are able to apply the *i.h.* on the facts that $\mathtt{C}'[p] \in \mathcal{C}^h_\vartheta$ and $\mathtt{C}'[\![x]\!] \notin \mathcal{C}^h_\vartheta$ to conclude that $\mathtt{C}' \in Y^\vartheta$. This in turn implies that $\mathtt{C}'[y\backslash s] \in Y^\vartheta$ by the fact that adding an arbitrary substitution preserves evaluation contexts (Lem. 80).
   (b) **The hole of $\mathtt{C}$ is to the right, *i.e.* $\mathtt{C} = t[y\backslash \mathtt{C}']$.** This case is not possible, since $t \in \mathcal{C}^h_\vartheta$ and $y \notin \mathsf{ngv}(t)$ by formation, and this implies that $t[y\backslash \mathtt{C}'[\![x]\!]] \in \mathcal{C}^h_\vartheta$, contradicting the hypothesis that $\mathtt{C}[\![x]\!] \notin \mathcal{C}^h_\vartheta$.
5. **nfSub,** $\mathtt{C}[p] = t[y\backslash M^\vartheta] \in \mathbb{X}_\vartheta$ **with** $t \in \mathbb{X}_{\vartheta\cup\{y\}}$ **and** $M^\vartheta \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$**.** Let us check that $\mathtt{C} \in Y^\vartheta$. If $\mathtt{C}$ is empty, *i.e.* $\mathtt{C} = \Box$, we trivially have $\mathtt{C} \in Y^\vartheta$. Otherwise, $\mathtt{C}$ is non-empty and there are two possibilities:
   (a) **The hole of $\mathtt{C}$ is to the left, *i.e.* $\mathtt{C} = \mathtt{C}'[y\backslash M^\vartheta]$.** Then $\mathtt{C}'[p] \in \mathbb{X}_{\vartheta\cup\{y\}}$ by formation. Moreover, we claim that $\mathtt{C}'[\![x]\!] \notin \mathbb{X}_{\vartheta\cup\{x\}}$. By contradiction, suppose that $\mathtt{C}'[\![x]\!] \in \mathbb{X}_{\vartheta\cup\{x\}}$. Then $\mathtt{C}'[\![x]\!][y\backslash M^\vartheta] \in \mathcal{C}^h_\vartheta$, contradicting the hypothesis that $\mathtt{C}[\![x]\!] \notin \mathcal{C}^h_\vartheta$.

So by *i.h.* we obtain that $\mathtt{C}' \in Y^{\vartheta\cup\{x\}}$ and, applying the context forming rule for structural substitutions (ESubLStr), we get $\mathtt{C}'[y\backslash M^{\vartheta}] \in Y^{\vartheta}$, that is to say $\mathtt{C} \in Y^{\vartheta}$, as required.

(b) **The hole of $\mathtt{C}$ is to the right, *i.e.* $\mathtt{C} = t[y\backslash \mathtt{C}']$.** Then $t \in \mathbb{X}_{\vartheta\cup\{y\}}$ and $y \in \mathsf{ngv}(t)$ by formation. This implies that $t = \mathtt{C}_1[\![y]\!]$ with $\mathtt{C}_1 \in Y^{\vartheta}$ by Lem. 82
Note that $\mathtt{C}'[p] = M^{\vartheta} \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$. Moreover, we claim that $\mathtt{C}'[\![x]\!] \notin \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$. By contradiction, suppose that $\mathtt{C}'[\![x]\!] \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$. Then $t[y\backslash \mathtt{C}'[\![x]\!]] \in \mathcal{C}^h_{\vartheta}$, contradicting the hypothesis that $\mathtt{C}[\![x]\!] \notin \mathcal{C}^h_{\vartheta}$. So by *i.h.* we have that $\mathtt{C}' \in \mathcal{C}^{\cdot}_{\vartheta}$.
Combining the facts that $\mathtt{C}_1 \in Y^{\vartheta}$ and $\mathtt{C}' \in \mathcal{C}^{\cdot}_{\vartheta}$, by applying the formation rule for evaluation contexts going inside substitutions, we conclude that $\mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}'] \in Y^{\vartheta}$, as required.

6. **nfStruct.** Suppose that $\mathtt{C}[p] \in \mathcal{N}_{\vartheta}$ by applying the rule requiring that $\mathtt{C}[p] \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$. We also know that $\mathtt{C}[\![x]\!] \notin \mathcal{N}_{\vartheta}$, so $\mathtt{C}[\![x]\!] \notin \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$, as the set of strong ϑ-normal forms contains the set of strong ϑ-structures. By *i.h.* we have that $\mathtt{C} \in \mathcal{C}^{\cdot}_{\vartheta}$, as desired.

$\square$

Definition 85 (Structural variables)**.** *Let $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$. The set of* structural variables *of $\mathtt{C}$, written* $\mathsf{sv}(\mathtt{C})$ *is defined by induction on the derivation that $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$ as follows:*

1. **EBox, $\mathtt{C} = \square \in \mathcal{C}^h_{\vartheta}$.**
$$\mathsf{sv}(\square) := \varnothing$$

2. **EAppL, $\mathtt{C} = \mathtt{C}_1\, t \in \mathcal{C}^h_{\vartheta}$ with $h \neq \lambda$.**
$$\mathsf{sv}(\mathtt{C}_1\, t) := \mathsf{sv}(\mathtt{C}_1)$$

3. **ESubLNonStr, $\mathtt{C} = \mathtt{C}_1[x\backslash t] \in \mathcal{C}^h_{\vartheta}$ with $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta}$, $x \notin \vartheta$, and $t \notin \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$.**
$$\mathsf{sv}(\mathtt{C}_1[x\backslash t]) := \mathsf{sv}(\mathtt{C}_1) \setminus \{x\}$$

4. **ESubLStr, $\mathtt{C} = \mathtt{C}_1[x\backslash t] \in \mathcal{C}^h_{\vartheta}$ with $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x\}}$, and $t \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$.**
$$\mathsf{sv}(\mathtt{C}_1[x\backslash t]) := (\mathsf{sv}(\mathtt{C}_1) \setminus \{x\}) \cup \begin{cases} \mathsf{ngv}(t) & \textit{if } x \in \mathsf{sv}(\mathtt{C}_1) \\ \varnothing & \textit{otherwise} \end{cases}$$

5. **ESubsR, $\mathtt{C} = \mathtt{C}_1[\![x]\!][x\backslash \mathtt{C}_2] \in \mathcal{C}^h_{\vartheta}$ with $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta}$ and $\mathtt{C}_2 \in \mathcal{C}^{\cdot}_{\vartheta}$.**
$$\mathsf{sv}(\mathtt{C}_1[\![x]\!][x\backslash \mathtt{C}_2]) := (\mathsf{sv}(\mathtt{C}_1) \setminus \{x\}) \cup \mathsf{sv}(\mathtt{C}_2)$$

6. **EAppRStr, $\mathtt{C} = t\, \mathtt{C}_1 \in \mathcal{C}^h_{\vartheta}$ with $t \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$ and $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta}$.**
$$\mathsf{sv}(t\, \mathtt{C}_1) := \mathsf{ngv}(t) \cup \mathsf{sv}(\mathtt{C}_1)$$

7. **ELam, $\mathtt{C} = \lambda x.\mathtt{C}_1 \in \mathcal{C}^{\lambda}_{\vartheta}$ with $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x\}}$.**
$$\mathsf{sv}(\lambda x.\mathtt{C}_1) := \mathsf{sv}(\mathtt{C}_1) \setminus \{x\}$$

8. **EAppRCons,** $t\,\mathtt{C} \in \mathcal{C}_\vartheta^{\mathsf{hc}(t)}$ **and** $t \in \mathcal{K}_\vartheta$ **and** $\mathtt{C} \in \mathcal{C}_\vartheta^h$ **.**

$$\mathsf{sv}(t\,\mathtt{C}_1) := \mathsf{ngv}(t) \cup \mathsf{sv}(\mathtt{C}_1)$$

9. **ECase1.** $\mathtt{case}\ \mathtt{C}_1\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{C}_\vartheta^{\cdot}$ **and** $\mathtt{C}_1 \in \mathcal{C}_\vartheta^h$ **and** $h \notin \{\mathbf{c}_i\}_{i\in I}$ **or** $h = \mathbf{c}_j \in \{\mathbf{c}_i\}_{i\in I}$ **and** $|\mathtt{A}(\mathtt{C}, y)| \neq |\bar{x_j}|$**. .**

$$\mathsf{sv}(\mathtt{case}\ \mathtt{C}_1\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}) := \mathsf{sv}(\mathtt{C}_1)$$

10. **ECase2.** $\mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}_1, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n \in \mathcal{C}_\vartheta^{\cdot}$ **and** $t \in \mathcal{N}_\vartheta$ **and** $t \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ **and** $t_k \in \mathcal{N}_{\vartheta\cup\bar{x_k}}$ **for all** $k < j$ **and** $\mathtt{C}_1 \in \mathcal{C}_{\vartheta\cup\bar{x_i}}^h$ **.**

$$\mathsf{sv}(\mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}_1, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n) := \mathsf{ngv}(t) \cup \bigcup_{i=1}^{j-1} \mathsf{ngv}(t_i) \cup \mathsf{sv}(\mathtt{C}_1) \setminus \{\bar{x_j}\}$$

Lemma 86 (Reachable and structural variables are frozen)**.** *The following properties hold:*

1. *Let $t \in \mathcal{C}_\vartheta^h$, where $\mathcal{C}_\vartheta^h$ is either $\mathcal{S}_\vartheta$ or $\mathcal{E}_\vartheta$ or $\mathcal{L}_\vartheta$ or $\mathcal{N}_\vartheta$. Then $\mathsf{ngv}(t) \subseteq \vartheta$.*
2. *Let $\mathtt{C} \in \mathcal{C}_\vartheta^h$. Then $\mathsf{sv}(\mathtt{C}) \subseteq \vartheta$.*

Proof. The proof of the first item is by straightforward induction on the derivation that $t \in \mathcal{C}_\vartheta^h$. The proof of the second item is by induction on the derivation that $\mathtt{C} \in \mathcal{C}_\vartheta^h$. There are only interesting two interesting cases, when $\mathtt{C}$ is formed by appending a structural substitution, and when it is formed by going to the right of an application:

1. **ESubLStr,** $\mathtt{C} = \mathtt{C}_1[x\backslash t]$ **with** $t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ **and** $\mathtt{C}_1 \in \mathcal{C}_{\vartheta\cup\{x\}}^h$**.** Then $\mathsf{sv}(\mathtt{C}) \subseteq (\mathsf{sv}(\mathtt{C}_1) \setminus \{x\}) \cup \mathsf{ngv}(t) \subseteq \vartheta$ since $\mathsf{sv}(\mathtt{C}_1) \subseteq \vartheta \cup \{x\}$ by the *i.h.*, and $\mathsf{ngv}(t) \subseteq \vartheta$ by the first item of this lemma.
2. **EAppRStr,** $\mathtt{C} = t\,\mathtt{C}_1$ **with** $t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ **and** $\mathtt{C}_1 \in \mathcal{C}_\vartheta^h$**.** Then $\mathsf{sv}(\mathtt{C}) = \mathsf{ngv}(t) \cup \mathsf{sv}(\mathtt{C}_1) \subseteq \vartheta$ since $\mathsf{ngv}(t) \subseteq \vartheta$ by the first item of this lemma, and $\mathsf{sv}(\mathtt{C}_1) \subseteq \vartheta$ by the *i.h.*.

□

Lemma 87 (Non-structural variables are not required in ϑ)**.** *If $\mathtt{C} \in \mathcal{C}_{\vartheta\cup\{x\}}^h$ and $x \notin \mathsf{sv}(\mathtt{C})$, then $\mathtt{C} \in \mathcal{C}_\vartheta^h$.*

Proof. By induction on the size of the context $\mathtt{C}$, and then by case analysis on the last step of the derivation that $\mathtt{C} \in \mathcal{C}_{\vartheta\cup\{x\}}^h$:

1. **EBox,** $\mathtt{C} = \Box \in \mathcal{C}_{\vartheta\cup\{x\}}^h$**.** Then $\Box \in \mathcal{C}_\vartheta^h$.
2. **EAppL,** $\mathtt{C} = \mathtt{C}_1\,t \in \mathcal{C}_{\vartheta\cup\{x\}}^h$ **with** $h \neq \lambda$ **and** $\mathtt{C}_1 \in \mathcal{C}_{\vartheta\cup\{x\}}^h$**.** Then $\mathsf{sv}(\mathtt{C}) = \mathsf{sv}(\mathtt{C}_1)$, so $x \notin \mathsf{sv}(\mathtt{C}_1)$. By *i.h.* $\mathtt{C}_1 \in \mathcal{C}_\vartheta^h$, so $\mathtt{C}_1\,t \in \mathcal{C}_\vartheta^h$.

3. **ESubLNonStr,** $\mathtt{C} = \mathtt{C}_1[y\backslash t] \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ **with** $t \notin \mathcal{S}_{\vartheta\cup\{x\}} \cup \mathcal{E}_{\vartheta\cup\{x\}}$**,** $y \notin \vartheta \cup \{x\}$**, and** $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x\}}$**.** Then $x \notin \mathsf{sv}(\mathtt{C}_1[x\backslash t]) \supseteq \mathsf{sv}(\mathtt{C}_1) \setminus \{y\}$, so $x \notin \mathsf{sv}(\mathtt{C}_1) \setminus \{y\}$. Observe that $x \neq y$ by the variable convention, so actually $x \notin \mathsf{sv}(\mathtt{C}_1)$. By *i.h.* $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta}$. By the fact that adding an arbitrary substitution preserves evaluation contexts (Lem. 80) we obtain that $\mathtt{C}_1[x\backslash t] \in \mathcal{C}^h_{\vartheta}$, as required.

4. **ESubLStr,** $\mathtt{C} = \mathtt{C}_1[y\backslash t] \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ **with** $t \in \mathcal{S}_{\vartheta\cup\{x\}} \cup \mathcal{E}_{\vartheta\cup\{x\}}$ **and** $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x,y\}}$**.** Then $x \notin \mathsf{sv}(\mathtt{C}_1[y\backslash t]) \supseteq \mathsf{sv}(\mathtt{C}_1) \setminus \{y\}$, so $x \notin \mathsf{sv}(\mathtt{C}_1) \setminus \{y\}$. Observe that $x \neq y$ by the variable convention, so actually $x \notin \mathsf{sv}(\mathtt{C}_1)$. Hence we can apply the *i.h.*, obtaining that $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{y\}}$. We consider two cases, depending on whether y is structural in $\mathtt{C}_1$:

   (a) **If** $y \in \mathsf{sv}(\mathtt{C}_1)$**.** Then, by definition of the structural variables, we have that $\mathsf{sv}(\mathtt{C}) = (\mathsf{sv}(\mathtt{C}_1) \setminus \{y\}) \cup \mathsf{ngv}(t)$. In particular, $x \notin \mathsf{ngv}(t)$. By the fact that unreachable variables are not required in "ϑ" (Lem. 62) we have that $t \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$. Now we can apply the formation rule for contexts adding a structural substitution (ESubLStr), and conclude $\mathtt{C}_1[y\backslash t] \in \mathcal{C}^h_{\vartheta}$, as required.

   (b) **If** $y \notin \mathsf{sv}(\mathtt{C}_1)$**.** Then we may apply the *i.h.* again on the fact that $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{y\}}$ to obtain that $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta}$. By the fact that adding an arbitrary substitution preserves evaluation contexts (Lem. 80) we have that $\mathtt{C}_1[x\backslash t] \in \mathcal{C}^h_{\vartheta}$, as required.

5. **ESubsR,** $\mathtt{C} = \mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}_2] \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ **with** $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ **and** $\mathtt{C}_2 \in \mathcal{C}^{\cdot}_{\vartheta\cup\{x\}}$**.** Then $x \notin \mathsf{sv}(\mathtt{C}) = (\mathsf{sv}(\mathtt{C}_1) \setminus \{y\}) \cup \mathsf{sv}(\mathtt{C}_2)$. Moreover, $x \neq y$ by the variable convention, so we know that $x \notin \mathsf{sv}(\mathtt{C}_1)$ and that $x \notin \mathtt{C}_2$. We may apply the *i.h.* on both $\mathtt{C}_1$ and $\mathtt{C}_2$ to get that $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta}$ and $\mathtt{C}_2 \in \mathsf{NACtxt}_{\vartheta}$, which imply $\mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}_2] \in \mathcal{C}^h_{\vartheta}$.

6. **EAppRStr,** $\mathtt{C} = t\,\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ **with** $t \in \mathcal{S}_{\vartheta\cup\{x\}} \cup \mathcal{E}_{\vartheta\cup\{x\}}$ **and** $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x\}}$**.** Then $x \notin \mathsf{sv}(\mathtt{C}) = \mathsf{ngv}(t) \cup \mathsf{sv}(\mathtt{C}_1)$, so $x \notin \mathsf{ngv}(t)$ and $x \notin \mathsf{sv}(\mathtt{C}_1)$. By the fact that unreachable variables are not required in "ϑ" (Lem. 62) we have that $t \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$. By *i.h.* we have that $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta}$. So $t\,\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta}$ as required.

7. **ELam,** $\mathtt{C} = \lambda y.\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ **with** $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x,y\}}$**.** Then $x \notin \mathsf{sv}(\mathtt{C}) = \mathsf{sv}(\mathtt{C}_1) \setminus \{y\}$. Observe that $x \neq y$ by the variable convention, so actually $x \notin \mathsf{sv}(\mathtt{C}_1)$. By *i.h.* we obtain that $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{y\}}$, so $\lambda y.\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta}$ and we conclude.

8. **EAppRCons,** $t\,\mathtt{C} \in \mathcal{C}^{\mathsf{hc}(t)}_{\vartheta\cup\{x\}}$ **and** $t \in \mathcal{K}_{\vartheta}$ **and** $\mathtt{C} \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ **.** Same as the case EAppRStr.

9. **ECase1.** $\mathtt{case}\ \mathtt{C}_1\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{C}^{\cdot}_{\vartheta\cup\{x\}}$ **and** $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ **and** $h \notin \{\mathbf{c}_i\}_{i\in I}$ **or** $h = \mathbf{c}_j \in \{\mathbf{c}_i\}_{i\in I}$ **and** $|\mathtt{A}(\mathtt{C}, y)| \neq |\bar{x_j}|$ **.** Then $\mathsf{sv}(\mathtt{case}\ \mathtt{C}_1\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{C}^{\cdot}_{\vartheta}) := \mathsf{sv}(\mathtt{C}_1)$, so $x \notin \mathsf{sv}(\mathtt{C}_1)$. By *i.h.* $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta}$, so $\mathtt{case}\ \mathtt{C}_1\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{C}^{\cdot}_{\vartheta}$.

10. **ECase2.** $\mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}_1, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n \in \mathcal{C}^{\cdot}_{\vartheta\cup\{x\}}$ **and** $t \in \mathcal{N}_{\vartheta}$ **and** $t \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ **and** $t_k \in \mathcal{N}_{\vartheta\cup\bar{x_k}}$ **for all** $k < j$ **and** $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x\}\cup\bar{x_i}}$ **.** Then $x \notin$ $sv\mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}_1, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n \in \mathcal{C}^{\cdot}_{\vartheta} := \mathsf{sv}(\mathtt{C}_1) \setminus \{\bar{x_j}\}$. We reason as above, making use of the *i.h.* and fact that unreachable variables are not required in "ϑ" (Lem. 62).

□

Lemma 88 (Structural variables are below evaluation contexts)**.** *Let* $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ *where* $x \in \mathsf{sv}(\mathtt{C}_1)$ *and* $x \notin \vartheta$. *Then for any term* p *there is a context* $\mathtt{C}_2 \in \mathcal{C}^h_{\vartheta}$ *such that* $\mathtt{C}_1[p] = \mathtt{C}_2\llbracket x\rrbracket$.

Proof. By induction on the size of the term $\mathtt{C}_1[p]$, and then by case analysis on the last step of the derivation that $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x\}}$.

1. **EBox,** $\mathtt{C}_1 = \Box \in \mathcal{C}^h_{\vartheta\cup\{x\}}$**.** Impossible, since $x \in \mathsf{sv}(\mathtt{C}_1) = \varnothing$.

2. **EAppL,** $\mathtt{C}_1 = \mathtt{C}_{11}\, t \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ **with** $\mathtt{C}_{11} \in \mathcal{C}^{\cdot}_{\vartheta\cup\{x\}}$**.** Then $\mathsf{sv}(\mathtt{C}_{11}\, t) = \mathsf{sv}(\mathtt{C}_{11})$, so by *i.h.* there exists a context $\mathtt{C}_{21} \in \mathcal{C}^{\cdot}_{\vartheta}$ such that $\mathtt{C}_{11}[p] = \mathtt{C}_{21}\llbracket x\rrbracket$. Then $\mathtt{C}_1[p] = \mathtt{C}_{11}[p]\, t = \mathtt{C}_{21}\llbracket x\rrbracket\, t$ and $\mathtt{C}_{21}\, t \in \mathcal{C}^h_{\vartheta}$.

3. **ESubLNonStr,** $\mathtt{C}_1 = \mathtt{C}_{11}[y\backslash t] \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ **with** $t \notin \mathcal{S}_{\vartheta\cup\{x\}} \cup \mathcal{E}_{\vartheta\cup\{x\}}$**,** $y \notin \vartheta \cup \{x\}$**, and** $\mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta\cup\{x\}}$**.** Then $\mathsf{sv}(\mathtt{C}_{11}[y\backslash t]) = \mathsf{sv}(\mathtt{C}_{11}) \setminus \{y\}$. In particular, $x \in \mathsf{sv}(\mathtt{C}_{11})$, so by *i.h.* there exists a context $\mathtt{C}_{21} \in \mathcal{C}^h_{\vartheta}$ such that $\mathtt{C}_{11}[p] = \mathtt{C}_{21}\llbracket x\rrbracket$. So $\mathtt{C}_1[p] = \mathtt{C}_{11}[p][y\backslash t] = \mathtt{C}_{21}\llbracket x\rrbracket[y\backslash t]$. By the fact that adding an arbitrary substitution preserves evaluation contexts (Lem. 80) we have that $\mathtt{C}_{21}[y\backslash t] \in \mathcal{C}^h_{\vartheta}$, as required.

4. **ESubLStr,** $\mathtt{C}_1 = \mathtt{C}_{11}[y\backslash t] \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ **with** $t \in \mathcal{S}_{\vartheta\cup\{x\}} \cup \mathcal{E}_{\vartheta\cup\{x\}}$ **and** $\mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta\cup\{x,y\}}$**.** We consider two cases, depending on whether x is a structural variable in $\mathtt{C}_{11}$:

   (a) **If** $x \in \mathsf{sv}(\mathtt{C}_{11})$**.** Then by *i.h.* there is a context $\mathtt{C}_{21} \in \mathcal{C}^h_{\vartheta\cup\{y\}}$ such that $\mathtt{C}_{11}[p] = \mathtt{C}_{21}\llbracket x\rrbracket$. We consider two further subcases, depending on whether y is a structural variable in $\mathtt{C}_{21}$:

      i. **If** $y \in \mathsf{sv}(\mathtt{C}_{21})$**.** We consider two more cases, depending on whether x is reachable in the structure t:

         A. **If** $x \in \mathsf{ngv}(t)$**.** Since $y \in \mathsf{sv}(\mathtt{C}_{21})$ we may apply the *i.h.* again, to obtain that there exists a context $\mathtt{C}_{31} \in \mathcal{C}^h_{\vartheta}$ such that $\mathtt{C}_{21}\llbracket x\rrbracket = \mathtt{C}_{31}\llbracket y\rrbracket$. Note that we are able to apply the *i.h.* since the term $\mathtt{C}_{21}\llbracket x\rrbracket = \mathtt{C}_{11}[p]$ is smaller than the original term, namely $\mathtt{C}_1[p] = \mathtt{C}_{11}[p][y\backslash t]$.
         By the fact that reachable variables are below evaluation contexts (Lem. 82) and $t \in \mathcal{S}_{\vartheta\cup\{x\}} \cup \mathcal{E}_{\vartheta\cup\{x\}}$ we know that there exists a context $\mathtt{C}_{22} \in \mathcal{C}^{\cdot}_{\vartheta}$ such that $t = \mathtt{C}_{22}\llbracket x\rrbracket$. So we know that

$$\begin{array}{rcll} \mathtt{C}_1[p] & = & \mathtt{C}_{11}[p][y\backslash t] & \\ & = & \mathtt{C}_{11}[p][y\backslash \mathtt{C}_{22}\llbracket x\rrbracket] & \\ & = & \mathtt{C}_{21}\llbracket x\rrbracket[y\backslash \mathtt{C}_{22}\llbracket x\rrbracket] & \\ & = & \mathtt{C}_{31}\llbracket y\rrbracket[y\backslash \mathtt{C}_{22}\llbracket x\rrbracket] & \text{with } \mathtt{C}_{31} \in \mathcal{C}^h_{\vartheta} \text{ and } \mathtt{C}_{22} \in \mathsf{NACtxt}_{\vartheta} \end{array}$$

         where, by applying the rule for building evaluation contexts by going inside substitutions (ESubsR), we have that $\mathtt{C}_{31}\llbracket y\rrbracket[y\backslash \mathtt{C}_{22}] \in \mathcal{C}^h_{\vartheta}$.

         B. **If** $x \notin \mathsf{ngv}(t)$**.** By the fact that unreachable variables are not needed in "ϑ" (Lem. 62) we have that $t \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$. So:

$$\begin{array}{rcll} \mathtt{C}_1[p] & = & \mathtt{C}_{11}[p][y\backslash t] & \\ & = & \mathtt{C}_{21}\llbracket x\rrbracket[y\backslash t] & \text{with } \mathtt{C}_{21} \in \mathcal{C}^h_{\vartheta\cup\{y\}} \text{ and } t \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta} \end{array}$$

         where, by applying the rule for building evaluation contexts with structural substitutions (ESubLStr), we have that $\mathtt{C}_{21}[y\backslash t] \in \mathcal{C}^h_{\vartheta}$.

ii. **If $y \notin \mathsf{sv}(\mathtt{C}_{21})$.** Then by the fact that non-structural variables are not required in "ϑ" (Lem. 87) we conclude that $\mathtt{C}_{21} \in \mathcal{C}^h_\vartheta$. So:

$$\begin{array}{rcl} \mathtt{C}_1[p] & = & \mathtt{C}_{11}[p][y\backslash t] \\ & = & \mathtt{C}_{21}[\![x]\!][y\backslash t] \quad \text{with } \mathtt{C}_{21} \in \mathcal{C}^h_\vartheta \end{array}$$

By the fact that adding an arbitrary substitution preserves evaluation contexts (Lem. 80), we have $\mathtt{C}_{21}[y\backslash t] \in \mathcal{C}^h_\vartheta$, as required.

(b) **If $x \notin \mathsf{sv}(\mathtt{C}_{11})$.** Recall that, by hypothesis, $x \in \mathsf{sv}(\mathtt{C}_1) = \mathsf{sv}(\mathtt{C}_{11}[y\backslash t])$ and that by definition of structural variables:

$$\mathsf{sv}(\mathtt{C}_{11}[y\backslash t]) = \mathsf{sv}(\mathtt{C}_{11}) \cup \begin{cases} \mathsf{ngv}(t) & \text{if } y \in \mathsf{sv}(\mathtt{C}_{11}) \\ \varnothing & \text{otherwise} \end{cases}$$

so, since $x \notin \mathsf{sv}(\mathtt{C}_{11})$, we must have that $y \in \mathsf{sv}(\mathtt{C}_{11})$ and $x \in \mathsf{ngv}(t)$.

Now, by applying the lemma that non-structural variables are not required in "ϑ" (Lem. 87) on the fact that $\mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta\cup\{x,y\}}$ we have $\mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta\cup\{y\}}$. Since $y \in \mathsf{sv}(\mathtt{C}_{11})$, by the *i.h.* we have that there exists a context $\mathtt{C}_{21} \in \mathcal{C}^h_\vartheta$ such that $\mathtt{C}_{11}[p] = \mathtt{C}_{21}[\![y]\!]$. Moreover, $x \in \mathsf{ngv}(t)$, so by the fact that reachable variables are below evaluation contexts (Lem. 82) we know that there exists a context $\mathtt{C}_{22} \in \mathcal{C}^{\cdot}_\vartheta$ such that $t = \mathtt{C}_{22}[\![x]\!]$. So we have:

$$\begin{array}{rcl} \mathtt{C}_1[p] & = & \mathtt{C}_{11}[p][y\backslash t] \\ & = & \mathtt{C}_{21}[\![y]\!][y\backslash t] \\ & = & \mathtt{C}_{21}[\![y]\!][y\backslash \mathtt{C}_{22}[\![x]\!]] \quad \text{with } \mathtt{C}_{21} \in \mathcal{C}^h_\vartheta \text{ and } \mathtt{C}_{22} \in \mathcal{C}^{\cdot}_\vartheta \end{array}$$

By applying the rule for building evaluation contexts by going inside substitutions (ESubsR), we obtain that $\mathtt{C}_{21}[\![y]\!][y\backslash \mathtt{C}_{22}] \in \mathcal{C}^h_\vartheta$, as required.

5. **ESubsR,** $\mathtt{C}_1 = \mathtt{C}_{11}[\![y]\!][y\backslash \mathtt{C}_{12}] \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ **with** $\mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ **and** $\mathtt{C}_{12} \in \mathsf{NACtxt}_{\vartheta\cup\{x\}}$**.** Then $\mathsf{sv}(\mathtt{C}_1) = \mathsf{sv}(\mathtt{C}_{11}[\![y]\!][y\backslash \mathtt{C}_{12}]) = (\mathsf{sv}(\mathtt{C}_{11}) \setminus \{y\}) \cup \mathsf{sv}(\mathtt{C}_{12})$. Observe that $x \neq y$ by the variable convention. We consider two cases, depending on whether x is a structural variable in $\mathtt{C}_{11}$:

(a) **If $x \in \mathsf{sv}(\mathtt{C}_{11})$.** Then by *i.h.* there is a context $\mathtt{C}_{21} \in \mathcal{C}^h_\vartheta$ such that $\mathtt{C}_{11}[\![y]\!] = \mathtt{C}_{21}[\![x]\!]$. By the fact that adding an arbitrary substitution preserves evaluation contexts (Lem. 80), we obtain that $\mathtt{C}_{21}[y\backslash \mathtt{C}_{12}[p]] \in \mathcal{C}^h_\vartheta$, as required.

(b) **If $x \notin \mathsf{sv}(\mathtt{C}_{11})$.** Then by the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C}_{11} \in \mathcal{C}^h_\vartheta$. Moreover, it must be the case that $x \in \mathsf{sv}(\mathtt{C}_{12})$, so by *i.h.* we have that there is a context $\mathtt{C}_{22} \in \mathsf{NACtxt}_\vartheta$ such that $\mathtt{C}_{12}[p] = \mathtt{C}_{22}[\![x]\!]$. By applying the formation rule for evaluation contexts going inside substitutions (ESubsR) we conclude that $\mathtt{C}_{11}[\![y]\!][y\backslash \mathtt{C}_{22}] \in \mathcal{C}^h_\vartheta$, as required.

6. **EAppRStr,** $\mathtt{C}_1 = t\,\mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ **with** $t \in \mathcal{S}_{\vartheta\cup\{x\}} \cup \mathcal{E}_{\vartheta\cup\{x\}}$ **and** $\mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta\cup\{x\}}$**.** Then $\mathsf{sv}(\mathtt{C}_1) = \mathsf{ngv}(t) \cup \mathsf{sv}(\mathtt{C}_{11})$ We consider two cases, depending on whether x is reachable in t:

(a) **If $x \in \mathsf{ngv}(t)$.** Then by the fact that reachable variables are below evaluation contexts (Lem. 82), the structure t can be written as of the form $\mathtt{C}_{21}[\![x]\!]$, with $\mathtt{C}_{21} \in \mathsf{NACtxt}_\vartheta$. By applying the formation rule for evaluation contexts going to the left of an application (EAppL) we conclude that $\mathtt{C}_{21}\,\mathtt{C}_{11}[p] \in \mathcal{C}^{\cdot}_\vartheta$.

(b) **If $x \notin \mathsf{ngv}(t)$.** Then by the fact that unreachable variables are not required in "ϑ" (Lem. 62), we have that $t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$. Moreover, $x \in \mathsf{sv}(\mathtt{C}_{11})$, so by *i.h.* there must exist a context $\mathtt{C}_{21} \in \mathcal{C}^h_\vartheta$ such that $\mathtt{C}_{11}[p] = \mathtt{C}_{21}[\![x]\!]$. By applying the formation rule for evaluation contexts going to the right of a structure (EAppRStr) we conclude that $t\,\mathtt{C}_{21} \in \mathcal{C}^h_\vartheta$ as required.

7. **ELam, $\mathtt{C}_1 = \lambda y.\mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ with $\mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta\cup\{x,y\}}$.** Note that $x \neq y$ by the variable convention. By *i.h.* there must exist a context $\mathtt{C}_{21} \in \mathcal{C}^h_{\vartheta\cup\{y\}}$ such that $\mathtt{C}_{11}[p] = \mathtt{C}_{21}[\![x]\!]$, and $\lambda x.\mathtt{C}_{21} \in \mathcal{C}^h_\vartheta$ as required.
8. **EAppRCons, $t\,\mathtt{C} \in \mathcal{C}^{\mathsf{hc}(t)}_{\vartheta\cup\{x\}}$ and $t \in \mathcal{K}_\vartheta$ and $\mathtt{C} \in \mathcal{C}^h_{\vartheta\cup\{x\}}$.** Similar to EAppRStr.
9. **ECase1. $\mathtt{case}\ \mathtt{C}_1\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{C}^{\cdot}_{\vartheta\cup\{x\}}$ and $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x\}}$ and $h \notin \{\mathbf{c}_i\}_{i\in I}$ or $h = \mathbf{c}_j \in \{\mathbf{c}_i\}_{i\in I}$ and $|\mathtt{A}(\mathtt{C}, y)| \neq |\bar{x_j}|$.** Similar to EAppL.
10. **ECase2. $\mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \dots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}_1, \dots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n \in \mathcal{C}^{\cdot}_{\vartheta\cup\{x\}}$ and $t \in \mathcal{N}_\vartheta$ and $t \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ and $t_k \in \mathcal{N}_{\vartheta\cup\bar{x_k}}$ for all $k < j$ and $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{x\}\cup\bar{x_i}}$.** Similar to EAppRStr.

$\square$

Lemma 89 (Evaluation contexts are backwards-stable by substitutions)**.** *Let $\widehat{\mathtt{C}}$ be a two-hole context, $x \notin \vartheta$ a variable, v any value, and p be a term such that x is not bound by $\widehat{\mathtt{C}}[p, \square]$. Then:*

$$\widehat{\mathtt{C}}[\square, v] \in \mathcal{C}^h_\vartheta \quad \textit{implies} \quad \left(\widehat{\mathtt{C}}[\square, x] \in \mathcal{C}^h_\vartheta \quad \textit{or} \quad \widehat{\mathtt{C}}[p, \square] \in \mathcal{C}^h_\vartheta\right)$$

Proof. Let us write $\square$ and $\boxtimes$ to distinguish the two holes of $\widehat{\mathtt{C}}$.

Remark. As an example that the left member of the disjunction does not always hold, consider the two-hole context $\widehat{\mathtt{C}} = (z\,\square)[z\backslash y\,\boxtimes]$ with $\vartheta = \{y\}$ and note that $\widehat{\mathtt{C}}[\square, I] = (z\,\square)[z\backslash y\,I]$ is a generalized $\{y\}$-evaluation context, since z is bound to a strong $\{y\}$-structure, but $\widehat{\mathtt{C}}[\square, x] = (z\,\square)[z\backslash y\,x]$ is not a generalized $\{y\}$-evaluation context, since z is bound to $y\,x$, which is not a strong $\{y\}$-structure. In such a situation, evaluation should focus on x, that is, the right member of the disjunction holds, and $\widehat{\mathtt{C}}[p, \square] = (z\,p)[z\backslash y\,\square]$ is a $\{y\}$-evaluation context. *(End of remark).*

The proof goes by induction on the derivation that $\widehat{\mathtt{C}}[\square, v]$ is an evaluation context over ϑ.

1. **EBox, $\widehat{\mathtt{C}}[\square, v] = \square \in \mathcal{C}^h_\vartheta$.** Impossible, as $\widehat{\mathtt{C}}[\square, v]$ must contain a value v as a subterm.
2. **EAppL, $\widehat{\mathtt{C}}[\square, v] = \mathtt{I}^\vartheta\, t \in \mathcal{C}^h_\vartheta$ with $\mathtt{I}^\vartheta \in \mathcal{C}^{\cdot}_\vartheta$.** If the value v is inside t, *i.e.* $\widehat{\mathtt{C}}[\square, \boxtimes] = \mathtt{I}^\vartheta\mathtt{C}[\boxtimes]$ then the left branch of the disjunction holds as $\mathtt{I}^\vartheta\mathtt{C}[x] \in \mathcal{C}^h_\vartheta$.
   Otherwise, the value v is inside $\mathtt{I}^\vartheta$, *i.e.* there is a two-hole context $\widehat{\mathtt{C}}_1$ such that $\widehat{\mathtt{C}}[\square, \boxtimes] = \widehat{\mathtt{C}}_1[\square, \boxtimes]\,t$ and $\widehat{\mathtt{C}}_1[\square, v] = \mathtt{I}^\vartheta \in \mathcal{C}^{\cdot}_\vartheta$. By *i.h.* there are two possibilities:
   - **the left branch holds**, *i.e.* $\widehat{\mathtt{C}}_1[\square, x] \in \mathcal{C}^{\cdot}_\vartheta$ and then the left branch holds again as $\widehat{\mathtt{C}}_1[\square, x]\,t \in \mathcal{C}^h_\vartheta$; or
   - **the right branch holds**, *i.e.* $\widehat{\mathtt{C}}_1[p, \square] \in \mathcal{C}^{\cdot}_\vartheta$ holds and then the right branch holds again as $\widehat{\mathtt{C}}_1[p, \square]\,t \in \mathcal{C}^h_\vartheta$.

3. **ESubLNonStr,** $\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}[y\backslash t] \in \mathcal{C}^h_\vartheta$ **where** $\mathtt{C} \in \mathcal{C}^h_\vartheta$ **and** $t \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$**.** If the value v is inside t, *i.e.* $\widehat{\mathtt{C}}[\Box, \boxtimes] = \mathtt{C}[y\backslash \mathtt{C}[\boxtimes]]$, we may apply the fact that adding an arbitrary substitution preserves evaluation contexts (Lem. 80), obtaining that $\mathtt{C}[y\backslash \mathtt{C}[x]] \in \mathcal{C}^h_\vartheta$, that is $\widehat{\mathtt{C}}[\Box, x] \in \mathcal{C}^h_\vartheta$ and the left branch of the disjunction holds.

   Otherwise the value v is inside $\mathtt{C}$, *i.e.* there is a two-hole context $\widehat{\mathtt{C}}_1$ such that $\widehat{\mathtt{C}}[\Box, \boxtimes] = \widehat{\mathtt{C}}_1[\Box, \boxtimes][y\backslash t]$ and $\widehat{\mathtt{C}}_1[\Box, v] = \mathtt{C} \in \mathcal{C}^h_\vartheta$. By *i.h.* there are two possibilities:

   - **the left branch holds**, *i.e.* $\widehat{\mathtt{C}}_1[\Box, x] \in \mathcal{C}^h_\vartheta$, and then the left branch holds again, as $\widehat{\mathtt{C}}_1[\Box, x][y\backslash t] \in \mathcal{C}^h_\vartheta$; or
   - **the right branch holds**, *i.e.* $\widehat{\mathtt{C}}_1[p, \Box] \in \mathcal{C}^h_\vartheta$, and then the right branch holds again, as $\widehat{\mathtt{C}}_1[p, \Box][y\backslash t] \in \mathcal{C}^h_\vartheta$.

4. **ESubLStr,** $\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}[y\backslash M^\vartheta] \in \mathcal{C}^h_\vartheta$ **where** $\mathtt{C} \in \mathbb{X}_{\vartheta\cup\{y\}}$ **and** $M^\vartheta \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$**.** If the value v is inside M^ϑ, *i.e.* $M^\vartheta = \mathtt{C}[v]$, we consider two further subcases, depending on whether $\mathtt{C}[\![x]\!]$ is a strong ϑ-structure:

   (a) **If** $\mathtt{C}[\![x]\!] \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$**.** Applying the formation rule for generalized ϑ-evaluation contexts, using a structural substitution (ESubLStr), we conclude that $\mathtt{C}[y\backslash \mathtt{C}[\![x]\!]] \in \mathcal{C}^h_\vartheta$, that is $\widehat{\mathtt{C}}[\Box, x] \in \mathcal{C}^h_\vartheta$, and the left branch of the disjunction holds.

   (b) **If** $\mathtt{C}[\![x]\!] \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$**.** Then, since $\mathtt{C}[v] \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ but $\mathtt{C}[\![x]\!] \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$, we have that $\mathtt{C}$ is $\mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$-critical. By Lem. 84 we know that every $\mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$-critical context is a $\mathcal{C}^{\cdot}_\vartheta$ context, so $\mathtt{C} \in \mathcal{C}^{\cdot}_\vartheta$.

   We consider two further subcases, depending on whether y is a structural variable in $\mathtt{C}$:

   i. **If** $y \in \mathsf{sv}(\mathtt{C})$**.** Then by the fact that structural variables are below evaluation contexts (Lem. 88) there is a context $\mathtt{C}_2 \in \mathcal{C}^h_\vartheta$ such that $\mathtt{C}[p] = \mathtt{C}_2[\![y]\!]$. This means that $\mathtt{C}[p][y\backslash \mathtt{C}] = \mathtt{C}_2[\![y]\!][y\backslash \mathtt{C}] \in \mathcal{C}^h_\vartheta$ since $\mathtt{C} \in \mathcal{C}^{\cdot}_\vartheta$ is a non-answer context. So the right branch holds.

   ii. **If** $y \notin \mathsf{sv}(\mathtt{C})$**.** Then by the fact that non-structural variables are not required in "ϑ" (Lem. 87) we know that $\mathtt{C} \in \mathcal{C}^h_\vartheta$. By the fact that adding an arbitrary substitution preserves evaluation contexts (Lem. 80), we conclude that $\mathtt{C}[y\backslash \mathtt{C}[\![x]\!]] \in \mathcal{C}^h_\vartheta$, and the left branch holds.

   Otherwise, the value v is inside $\mathtt{C}$, *i.e.* there is a two-hole context $\widehat{\mathtt{C}}_1$ such that $\widehat{\mathtt{C}}[\Box, \boxtimes] = \widehat{\mathtt{C}}_1[\Box, \boxtimes][x\backslash M^\vartheta]$ and $\widehat{\mathtt{C}}_1[\Box, v] = \mathtt{C} \in \mathbb{X}_{\vartheta\cup\{y\}}$. By *i.h.* there are two possibilities:

   - **the left branch holds**, *i.e.* $\widehat{\mathtt{C}}_1[\Box, x] \in \mathbb{X}_{\vartheta\cup\{y\}}$, and then the left branch holds again, as $\widehat{\mathtt{C}}_1[\Box, x][y\backslash M^\vartheta] \in \mathcal{C}^h_\vartheta$; or
   - **the right branch holds**, *i.e.* $\widehat{\mathtt{C}}_1[p, \Box] \in \mathbb{X}_{\vartheta\cup\{y\}}$, and then the right branch holds again, as $\widehat{\mathtt{C}}_1[p, \Box][y\backslash M^\vartheta] \in \mathcal{C}^h_\vartheta$.

5. **ESubsR,** $\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}[\![y]\!][y\backslash \mathtt{I}^\vartheta] \in \mathcal{C}^h_\vartheta$ **with** $\mathtt{C} \in \mathcal{C}^h_\vartheta$ **and** $\mathtt{I}^\vartheta \in \mathcal{C}^{\cdot}_\vartheta$**.** If the value v is inside $\mathtt{C}[\![y]\!]$, there are two cases, depending on whether the hole of $\mathtt{C}$ lies inside the value v or not:

   (a) **If the hole of $\mathtt{C}$ lies inside v.** Then $\mathtt{C} = \mathtt{C}_1[\lambda z.\mathtt{C}_2]$ where $v = \lambda z.\mathtt{C}_2[\![y]\!]$ and $\widehat{\mathtt{C}}[\Box, \boxtimes] = \mathtt{C}_1[\boxtimes][y\backslash \mathtt{I}^\vartheta[\Box]]$. By the decomposition of evaluation contexts lemma (Lem. 50) we know that $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$. By the fact that adding an arbitrary substitution preserves evaluation contexts (Lem. 80), $\mathtt{C}_1[x\backslash \mathtt{I}^\vartheta[p]] \in \mathcal{C}^h_\vartheta$. This means that $\widehat{\mathtt{C}}[p, \Box] = \mathtt{C}_1[x\backslash \mathtt{I}^\vartheta[p]] \in \mathcal{C}^h_\vartheta$, so the right branch holds.

(b) **If the hole of $\mathtt{C}$ and the position of v are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}_1$ such that $\widehat{\mathtt{C}}_1[\Box, v] = \mathtt{C}$. Note, in particular, that $\widehat{\mathtt{C}}_1[y, v] = \mathtt{C}[\![y]\!]$, and $\widehat{\mathtt{C}}[\Box, \boxtimes] = \widehat{\mathtt{C}}_1[y, \boxtimes][y \backslash \mathtt{I}^{\vartheta}[\Box]]$. By *i.h.* there are two possibilities:

   - **the left branch holds**, *i.e.* $\widehat{\mathtt{C}}_1[\Box, x] \in \mathcal{C}^h_{\vartheta}$, and then the left branch holds again, as $\widehat{\mathtt{C}}[\Box, x] = \widehat{\mathtt{C}}_1[y, x][y \backslash \mathtt{I}^{\vartheta}] \in \mathcal{C}^h_{\vartheta}$; or
   - **the right branch holds**, *i.e.* $\widehat{\mathtt{C}}_1[y, \Box] \in \mathcal{C}^h_{\vartheta}$. Using the fact that adding an arbitrary substitution preserves evaluation contexts (Lem. 80), this implies that $\widehat{\mathtt{C}}[p, \Box] = \widehat{\mathtt{C}}_1[y, \Box][y \backslash \mathtt{I}^{\vartheta}[p]] \in \mathcal{C}^h_{\vartheta}$, and the right branch holds again.

Otherwise, the value v is inside $\mathtt{I}^{\vartheta}$. This means that there is a two-hole context $\widehat{\mathtt{C}}_1$ such that $\widehat{\mathtt{C}} = \mathtt{C}[\![y]\!][y \backslash \widehat{\mathtt{C}}_1]$ and $\widehat{\mathtt{C}}_1[\Box, v] = \mathtt{I}^{\vartheta}$. By *i.h.* there are two possibilities:

- **the left branch holds**, *i.e.* $\widehat{\mathtt{C}}_1[\Box, x] \in \mathcal{C}^{\cdot}_{\vartheta}$, and then the left branch holds again, as $\widehat{\mathtt{C}}[\Box, x] = \mathtt{C}[\![y]\!][y \backslash \widehat{\mathtt{C}}_1[\Box, x]] \in \mathcal{C}^h_{\vartheta}$; or
- **the right branch holds**, *i.e.* $\widehat{\mathtt{C}}_1[p, \Box] \in \mathcal{C}^{\cdot}_{\vartheta}$, and then the right branch holds again, as $\widehat{\mathtt{C}}[p, \Box] = \mathtt{C}[\![y]\!][y \backslash \widehat{\mathtt{C}}_1[p, \Box]] \in \mathcal{C}^h_{\vartheta}$.

6. **EAppRStr, $\widehat{\mathtt{C}}[\Box, v] = u\,\mathtt{C} \in \mathcal{C}^h_{\vartheta}$, with $u \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$ and $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$.** If the value v is inside u *i.e.* $u = \mathtt{C}[v]$ and $\widehat{\mathtt{C}}[\Box, \boxtimes] = \mathtt{C}[\boxtimes]\,\mathtt{C}[\Box]$, we consider two further subcases, depending on whether $\mathtt{C}[\![x]\!]$ is a strong ϑ-structure:

   (a) **If $\mathtt{C}[\![x]\!] \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$.** Applying the formation rule for generalized ϑ-evaluation contexts, going to the right of a structure (EAppRStr), we conclude that $\widehat{\mathtt{C}}[\Box, x] = \mathtt{C}[\![x]\!]\,\mathtt{C} \in \mathcal{C}^h_{\vartheta}$, so the left branch holds.

   (b) **If $\mathtt{C}[\![x]\!] \notin \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$.** Then, since $\mathtt{C}[v] \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$ but $\mathtt{C}[\![x]\!] \notin \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$, we have that $\mathtt{C}$ is $\mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$-critical. By Lem. 84 we know that every $\mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$-critical context is a $\mathcal{C}^{\cdot}_{\vartheta}$ context, so $\mathtt{C} \in \mathcal{C}^{\cdot}_{\vartheta} h$. Applying the formation rule for generalized ϑ-evaluation contexts, going to the left of an application (EAppL), we conclude that $\widehat{\mathtt{C}}[p, \Box] = \mathtt{C}\,\mathtt{C}[p] \in \mathcal{C}^h_{\vartheta}$, so the right branch holds.

   Otherwise, the value v is inside $\mathtt{C}$, that is, there is a two-hole context $\widehat{\mathtt{C}}_1$ such that $\widehat{\mathtt{C}} = u\,\widehat{\mathtt{C}}_1$ and $\mathtt{C} = \widehat{\mathtt{C}}_1[\Box, v]$. By *i.h.* there are two possibilities:

   - **the left branch holds**, *i.e.* $\widehat{\mathtt{C}}_1[\Box, x] \in \mathcal{C}^h_{\vartheta}$ and the left branch holds again, as $u\,\widehat{\mathtt{C}}_1[\Box, x] \in \mathcal{C}^h_{\vartheta}$ by the rule EAppRStr; or
   - **the right branch holds**, *i.e.* $\widehat{\mathtt{C}}_1[p, \Box] \in \mathcal{C}^h_{\vartheta}$ and the right branch holds again, as $u\,\widehat{\mathtt{C}}_1[p, \Box] \in \mathcal{C}^h_{\vartheta}$ by the rule EAppRStr.

7. **ELam, $\widehat{\mathtt{C}}[\Box, v] = \lambda y.\mathtt{C} \in \mathcal{C}^h_{\vartheta}$ with $\mathtt{C} \in \mathcal{C}^h_{\vartheta \cup \{y\}}$.** Then there must be a two-hole context $\widehat{\mathtt{C}}_1$ such that $\widehat{\mathtt{C}} = \lambda y.\widehat{\mathtt{C}}_1$ and $\mathtt{C} = \widehat{\mathtt{C}}_1[\Box, v]$. By *i.h.* there are two possibilities:

   - **the left branch holds**, *i.e.* $\widehat{\mathtt{C}}_1[\Box, x] \in \mathcal{C}^h_{\vartheta \cup \{y\}}$ and the left branch holds again, as $\lambda y.\widehat{\mathtt{C}}_1[\Box, x] \in \mathcal{C}^h_{\vartheta}$; or
   - **the right branch holds**, *i.e.* $\widehat{\mathtt{C}}_1[p, \Box] \in \mathcal{C}^h_{\vartheta \cup \{y\}}$ and the right branch holds again, as $\lambda y.\widehat{\mathtt{C}}_1[p, \Box] \in \mathcal{C}^h_{\vartheta}$.

8. **EAppRCons, $\widehat{\mathtt{C}}[\Box, v] = t\,\mathtt{C} \in \mathcal{C}^{\mathsf{hc}(t)}_{\vartheta}$ and $t \in \mathcal{K}_{\vartheta}$ and $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$.** Similar to EAppRStr.

9. **ECase1.** $\widehat{\mathtt{C}}[\Box, v] = \mathtt{case}\ \mathtt{C}_1\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{C}^{\cdot}_{\vartheta}$ **and** $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta}$ **and** $h \notin \{\mathbf{c}_i\}_{i\in I}$ **or** $h = \mathbf{c}_j \in \{\mathbf{c}_i\}_{i\in I}$ **and** $|\mathtt{A}(\mathtt{C}, y)| \neq |\bar{x_j}|$**.** Similar to EAppL.

10. **ECase2.** $\widehat{\mathtt{C}}[\Box, v] = \mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}_1, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n \in \mathcal{C}^{\cdot}_{\vartheta}$ **and** $t \in \mathcal{N}_{\vartheta}$ **and** $t \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ **and** $t_k \in \mathcal{N}_{\vartheta\cup\bar{x_k}}$ **for all** $k < j$ **and** $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\bar{x_i}}$**.** Similar to EAppRStr.

$\Box$

Lemma 90 (Effect of appending a substitution context on ϑ)**.** *$\mathcal{C}\mathtt{L} \in \mathcal{C}^h_{\vartheta}$ if and only if $\mathcal{C} \in \mathcal{C}^h_{\vartheta'}$ where $\vartheta' := \mathsf{fz}^{\vartheta}(\mathtt{L})$.*

Proof. Straightforward by induction on $\mathtt{L}$. For each substitution node two cases must be considered, depending on whether it holds a strong ϑ-structure or not. $\Box$

Lemma 91 (Effect of permutation rules on ϑ)**.** *Let ϑ be a set of variables and $\vartheta' := \mathsf{fz}^{\vartheta}([x\backslash v]\mathtt{L})$. If $\mathcal{C} \in \mathcal{C}^h_{\vartheta'}$ and $\mathtt{L}$ is a substitution context such that* $\operatorname{dom}\mathtt{L}$ *has no variables in common with $\mathcal{C}$, then $\mathcal{C}[x\backslash v\mathtt{L}] \in \mathcal{C}^h_{\vartheta}$.*

Proof. By induction on the length of $\mathtt{L}$ let us show that $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$.

1. **Base case,** $\mathtt{L} = \Box$**.** Then $\vartheta' = \vartheta$ since v is not a strong ϑ-structure, so we are done.

2. **Induction,** $\mathtt{L} = \mathtt{L}'[y\backslash t]$**.** We consider two subcases, depending on whether t is a strong ϑ-structure or not:

   (a) **If t is a strong ϑ-structure.** Then $\vartheta' = \mathsf{fz}^{\vartheta}(\mathtt{L}) \cup \{y\}$. Note that y does not occur in $\mathtt{C}$, so in particular $y \notin \mathsf{sv}(\mathtt{C})$. By the fact that non-structural variables are not required in "ϑ" (Lem. 87) we obtain that $\mathtt{C} \in \mathcal{C}^h_{\mathsf{fz}^{\vartheta}(\mathtt{L})}$. Then by *i.h.* $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$.

   (b) **If t is not a strong ϑ-structure.** Then $\vartheta' = \mathsf{fz}^{\vartheta}(\mathtt{L})$ and we conclude by *i.h.*.

Finally, by adding an arbitrary substitution (Lem. 80) we have that $\mathtt{C}[x\backslash v\mathtt{L}] \in \mathcal{C}^h_{\vartheta}$, as required. $\Box$

Definition 92 (Chain context)**.** *The sets of (ϑ, x)-chain contexts, ranged over by $\mathscr{L}, \mathscr{L}'$, etc., are defined as follows:*

$$\frac{\vartheta' = \mathsf{fz}^{\vartheta}(\mathtt{L}_2) \quad \mathcal{C} \in \mathcal{C}^{\cdot}_{\vartheta'} \quad \mathtt{L}_1, \mathtt{L}_2 \textit{ are substitution contexts}}{\langle \mathtt{L}_1, x, \mathcal{C}, \mathtt{L}_2\rangle \textit{ is a } (\vartheta, x)\textit{-chain context}}$$

$$\frac{\vartheta' = \mathsf{fz}^{\vartheta}(\mathtt{L}) \quad \mathcal{C} \in \mathcal{C}^{\cdot}_{\vartheta'} \quad \mathtt{L} \textit{ is a substitution context} \quad \mathscr{L} \textit{ is a } (\vartheta' \cup \{y\}, x)\textit{-chain context}}{\langle \mathscr{L}, y, \mathcal{C}, \mathtt{L}\rangle \textit{ is a } (\vartheta, x)\textit{-chain context}}$$

Given a (ϑ, x)-chain context $\mathscr{L}$, its instantiation on two terms t_1, t_2, written $t_1\mathscr{L}\{t_2\}$, is defined inductively as follows:

$$\begin{array}{rcl} t_1\langle \mathtt{L}_1, x, \mathcal{C}, \mathtt{L}_2\rangle\{t_2\} & := & t_1\mathtt{L}_1[x\backslash \mathcal{C}[t_2]]\mathtt{L}_2 \\ t_1\langle \mathscr{L}, y, \mathcal{C}, \mathtt{L}\rangle\{t_2\} & := & (t_1\mathscr{L}\{y\})[y\backslash \mathcal{C}[t_2]]\mathtt{L} \end{array}$$

Sometimes we write $\mathscr{L}_x^\vartheta$ to stress that $\mathscr{L}$ is a (ϑ, x)-chain context. The number of rules required to build a chain context $\mathscr{L}$ is called the number of jumps *of $\mathscr{L}$.*

In informal terms, a (ϑ, x_1)-chain context interpreted as a two-hole context, *i.e.* $\Box_1\mathscr{L}\{\Box_2\}$, is of the form:

$$\begin{array}{ll} \Box_1 & \mathtt{L}_1[x_1\backslash\mathtt{C}_1[\![x_2]\!]] \\ & \mathtt{L}_2[x_2\backslash\mathtt{C}_2[\![x_3]\!]] \\ & \dots \\ & \mathtt{L}_i[x_i\backslash\mathtt{C}_i[\![x_{i+1}]\!]] \\ & \dots \\ & \mathtt{L}_{n-1}[x_{n-1}\backslash\mathtt{C}_{n-1}[\![x_n]\!]] \\ & \mathtt{L}_n[x_n\backslash\mathtt{C}_n[\![\Box_2]\!]] \\ & \mathtt{L}_{n+1} \end{array}$$

where $n \geq 1$, each $\mathtt{C}_i \in \mathcal{C}^{\cdot}_{\vartheta_i}$, and each ϑ_i is the set of available frozen variables according to the definition of generalized evaluation context:

$$\vartheta_n = \mathsf{fz}^\vartheta(\mathtt{L}_{n+1}) \qquad \vartheta_{i-1} = \mathsf{fz}^{\vartheta_i}(\mathtt{L}_i[x_i\backslash\mathtt{C}_i[\![x_{i+1}]\!]])$$

Lemma 93 (Weakening for chain contexts)**.** *If $\mathscr{L}$ is a (ϑ, x)-chain context, and $\vartheta \subseteq \vartheta'$ then $\mathscr{L}$ is a (ϑ', x)-chain context.*

Proof. By induction on the formation rules for chain contexts, using the weakening lemma for evaluation contexts (Lem. 64). $\Box$

Definition 94 (Adding substitutions to chain contexts)**.** *If $\mathtt{L}$ is a substitution context, $\vartheta' = \mathsf{fz}^\vartheta(\mathtt{L})$ and $\mathscr{L}$ is a (ϑ', x)-chain context then we write $\mathscr{L}\mathtt{L}$ for the (ϑ, x)-chain context defined as follows:*

1. $\langle \mathtt{L}_1, x, \mathcal{C}, \mathtt{L}_2\rangle\mathtt{L} := \langle \mathtt{L}_1, x, \mathcal{C}, \mathtt{L}_2\mathtt{L}\rangle$
2. $\langle \mathscr{L}', x, \mathcal{C}, \mathtt{L}'\rangle\mathtt{L} := \langle \mathscr{L}', x, \mathcal{C}, \mathtt{L}'\mathtt{L}\rangle$

Note that $t_1(\mathscr{L}\mathtt{L})\{t_2\} = (t_1\mathscr{L}\{t_2\})\mathtt{L}$.

Lemma 95 (Stripping substitutions from a context using chain contexts)**.** *Let $\mathtt{C} \in \mathcal{C}^h_\vartheta$ be an evaluation context. Suppose that $\mathtt{C}[t] = s\mathtt{L}$ where all the substitution nodes in the spine of $\mathtt{L}$ belong to the context $\mathtt{C}$ (rather than to the subterm t), that is, one of the following holds:*

- **A.** $\mathtt{C} = \mathcal{C}\mathtt{L}$ *and* $s = \mathcal{C}[t]$.
- **B.** $\mathtt{C} = s\mathtt{L}_1[x\backslash\mathcal{C}]\mathtt{L}_2$ *and* $\mathtt{L} = \mathtt{L}_1[x\backslash\mathcal{C}[t]]\mathtt{L}_2$.

Then in each case the following more precise conditions hold:

- **A.** *There is an evaluation context $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta'}$ where $\vartheta' = \mathsf{fz}^\vartheta(\mathtt{L})$ such that:*

$$\mathtt{C} = \mathtt{C}_1\mathtt{L} \qquad s = \mathtt{C}_1[t]$$

- **B.** *There is an evaluation context $\mathtt{C} \in \mathcal{C}^h_{\vartheta'}$ where $\vartheta' = \mathsf{fz}^\vartheta(\mathtt{L})$, and a (ϑ, x)-chain context $\mathscr{L}$ such that:*

$$\mathtt{C} = \mathtt{C}_1[\![x]\!]\mathscr{L}\{\Box\} \qquad \mathtt{L} = \Box\mathscr{L}\{t\}$$

Proof. We proceed by induction on $\mathtt{L}$.

- **Empty,** $\mathtt{L} = \Box$**.** Then case **B** is impossible, since the hypothesis **B** requires that $\mathtt{L}$ has at least one substitution. So case **A** applies, with $\vartheta' = \vartheta$ and $\mathtt{C}_1 = \mathtt{C}$.

- **Non-empty,** $\mathtt{L} = \mathtt{L}'[y\backslash u]$**.** In case **A**, $\mathtt{C} = \mathtt{CL}$. It suffices to note that, by the decomposition of evaluation contexts (Lem. 50), $\mathtt{C} \in \mathcal{C}^h_{\vartheta'}$ where $\vartheta' = \mathsf{fz}^\vartheta(\mathtt{L})$, so by taking $\mathtt{C}_1 := \mathtt{C}$ we conclude.

  In case **B** we consider two subcases, depending on whether the hole of $\mathtt{C}$ lies inside u or inside one of the substitutions in $\mathtt{L}'$.

  1. **If the hole of** $\mathtt{C}$ **lies inside** u**.** Then $\mathtt{C} = s\mathtt{L}'[y\backslash\mathtt{C}]$. Then since $\mathtt{C}$ is an evaluation context, it must be built using the ESubsR rule. Hence we have that $\mathtt{C} = \mathtt{C}_2[\![y]\!][y\backslash\mathtt{I}^\vartheta_3]$ where $\vartheta'' = \mathsf{fz}^\vartheta([y\backslash\mathtt{I}^\vartheta_3[t]])$ and the contexts are evaluation contexts, that is $\mathtt{C}_2 \in \mathcal{C}^h_{\vartheta''}$ and $\mathtt{I}^\vartheta_3 \in \mathcal{C}^{\cdot}_\vartheta$.
     Note that $\mathtt{C}_2 = s\mathtt{L}'$, so by *i.h.*, there are two possibilities for $\mathtt{C}_2$:
     (a) **A.** Then $\mathtt{C}_2 = \mathtt{C}_{21}\mathtt{L}'$ with $s = \mathtt{C}_2[\![y]\!]$. Let us take $\vartheta' := \vartheta'''$, with $\mathtt{C}_1 := \mathtt{C}_{21}$ and $\mathscr{L} := \langle\mathtt{L}', y, \mathtt{I}^\vartheta_3, \Box\rangle$. Then we have indeed that $\mathscr{L}$ is a (y,ϑ)-chain context and:
     $$\begin{array}{rcl} \mathtt{C} & = & \mathtt{C}_2[\![y]\!][y\backslash\mathtt{I}^\vartheta_3] \\ & = & \mathtt{C}_{21}[\![y]\!]\mathtt{L}'[y\backslash\mathtt{I}^\vartheta_3] \\ & = & \mathtt{C}_1[\![y]\!]\mathtt{L}'[y\backslash\mathtt{I}^\vartheta_3] \\ & = & \mathtt{C}_1[\![y]\!]\mathscr{L}\{\Box\} \end{array} \qquad \begin{array}{rcl} \mathtt{L} & = & \mathtt{L}'[y\backslash\mathtt{I}^\vartheta_3[t]] \\ & = & \Box\langle\mathtt{L}', y, \mathtt{I}^\vartheta_3, \Box\rangle\{t\} \\ & = & \Box\mathscr{L}\{t\} \end{array}$$
     (b) **B.** Then $\mathtt{C}_2 = \mathtt{C}_{21}[\![z]\!]\mathscr{L}'\{\Box\}$ with $\mathtt{L}' = \Box\mathscr{L}'\{y\}$, where $\mathscr{L}'$ is a (ϑ'', z)-chain context. Recall that $\vartheta'' = \mathsf{fz}^\vartheta([y\backslash\mathtt{I}^\vartheta_3[t]])$ so it can be that $y \in \vartheta''$ or that $y \notin \vartheta''$. In any case, by the weakening lemma for chain contexts (Lem. 93) we have that $\mathscr{L}'$ is a $(\vartheta'' \cup \{y\}, z)$-chain context.
     Let us take $\vartheta' := \vartheta'''$, with $\mathtt{C}_1 := \mathtt{C}_{21}$ and $\mathscr{L} := \langle\mathscr{L}', y, \mathtt{I}^\vartheta_3, \Box\rangle$. Then we have indeed that $\mathscr{L}$ is a (ϑ, z)-chain context and:
     $$\begin{array}{rcl} \mathtt{C} & = & \mathtt{C}_2[\![y]\!][y\backslash\mathtt{I}^\vartheta_3] \\ & = & \mathtt{C}_{21}[\![z]\!]\mathscr{L}'\{y\}[y\backslash\mathtt{I}^\vartheta_3] \\ & = & \mathtt{C}_1[\![z]\!]\mathscr{L}'\{y\}[y\backslash\mathtt{I}^\vartheta_3] \\ & = & \mathtt{C}_1[\![z]\!]\mathscr{L}\{\Box\} \end{array} \qquad \begin{array}{rcl} \mathtt{L} & = & \mathtt{L}'[y\backslash\mathtt{I}^\vartheta_3[t]] \\ & = & \Box\mathscr{L}'\{y\}[y\backslash\mathtt{I}^\vartheta_3[t]] \\ & = & \Box\langle\mathscr{L}', y, \mathtt{I}^\vartheta_3, \Box\rangle\{t\} \\ & = & \Box\mathscr{L}\{t\} \end{array}$$

  2. **If the hole of** $\mathtt{C}$ **lies inside** $\mathtt{L}'$**.** Then $\mathtt{C} = \mathtt{C}_2[y\backslash u]$ and $\mathtt{C}_2 = s\mathtt{L}'$ where the hole of $\mathtt{C}_2$ is inside $\mathtt{L}'$. By *i.h.* case **B** applies for $\mathtt{C}_2$ so we have that there exist an evaluation context $\mathtt{C}_{21} \in \mathcal{C}^h_{\vartheta'''}$ and a (ϑ'', z)-chain context $\mathscr{L}'$ such that:
     $$\mathtt{C}_2 = \mathtt{C}_{21}[\![z]\!]\mathscr{L}'\{\Box\} \qquad \mathtt{L}' = \Box\mathscr{L}'\{t\}$$
     Hence by taking $\vartheta' := \vartheta'''$ with contexts $\mathtt{C}_1 := \mathtt{C}_{21}$ and $\mathscr{L} := \mathscr{L}'[y\backslash u]$ we have:
     $$\begin{array}{rcl} \mathtt{C} & = & \mathtt{C}_2[y\backslash u] \\ & = & \mathtt{C}_{21}[\![z]\!]\mathscr{L}'\{\Box\}[y\backslash u] \\ & = & \mathtt{C}_1[\![z]\!]\mathscr{L}'\{\Box\}[y\backslash u] \\ & = & \mathtt{C}_1[\![z]\!](\mathscr{L}'[y\backslash u])\{\Box\} \\ & = & \mathtt{C}_1[\![z]\!]\mathscr{L}\{\Box\} \end{array} \qquad \begin{array}{rcl} \mathtt{L} & = & \mathtt{L}'[y\backslash u] \\ & = & \Box\mathscr{L}'\{t\}[y\backslash u] \\ & = & \Box(\mathscr{L}'[y\backslash u])\{t\} \\ & = & \Box\mathscr{L}\{t\} \end{array}$$

$\square$

Lemma 96 (Stripping substitutions from a $\mathtt{lsv}$ redex using chain contexts)**.** *If* $\mathtt{C}_1[\mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}']] = t\mathtt{L}$ *where* $\mathtt{C}_1[\mathtt{C}_2[x\backslash v\mathtt{L}']] \in \mathcal{C}^h_\vartheta$ *is an evaluation context then at least one of the following four possibilities holds:*

1. **A.** $\mathtt{C}_1 = \mathtt{C}_{11}\mathtt{L}$ *where* $\vartheta'' = \mathsf{fz}^\vartheta(\mathtt{L})$ *and* $\mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta''}$.
2. **B.** $\mathtt{C}_1 = \mathtt{C}_{11}[\![y]\!]\mathscr{L}\{\square\}$ *such that:*
$$\mathtt{L} = \square\mathscr{L}\{\mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}']\}$$
where $\vartheta'' = \mathsf{fz}^\vartheta(\mathtt{L})$*, the evaluation context* $\mathtt{C}_{11}$ *is in* $\mathcal{C}^h_{\vartheta''}$ *and* $\mathscr{L}$ *is a* (ϑ, y)*-chain context.*
3. **C.** $\mathtt{C}_2 = \mathtt{C}_{21}\tilde{\mathtt{L}}$ *such that:*
$$\mathtt{L} = \mathtt{C}_1[\tilde{\mathtt{L}}[x\backslash v\mathtt{L}']]$$
where $\vartheta'' = \mathsf{fz}^\vartheta(\tilde{\mathtt{L}})$*, the context* $\mathtt{C}_1$ *is a substitution context, and the evaluation context* $\mathtt{C}_{21}$ *is in* $\mathcal{C}^h_{\vartheta''}$.
4. **D.** $\mathtt{C}_2 = \mathtt{C}_{21}[\![y]\!]\{\square\}$ *such that:*
$$\mathtt{L} = \mathtt{C}_1[\square\mathscr{L}\{x\}[x\backslash v\mathtt{L}']]$$
where $\vartheta'' = \mathsf{fz}^\vartheta(\mathtt{L})$*, the context* $\mathtt{C}_1$ *is a substitution context, the evaluation context* $\mathtt{C}_{21}$ *is in* $\mathcal{C}^h_{\vartheta''}$*, and* $\mathscr{L}$ *is a* (ϑ'', y)*-chain context.*

Proof. We know that $\mathtt{C}_1[\mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}']] = t\mathtt{L}$. We consider two cases, depending on whether $\mathtt{L}$ is "contained" in $\mathtt{C}_1$, that is, all the substitution nodes in the spine of $\mathtt{L}$ belong to the context $\mathtt{C}_1$, or otherwise:

1. **If all the substitution nodes in the spine of $\mathtt{L}$ belong to the context $\mathtt{C}_1$.** That is, the substitution nodes in $\mathtt{L}$ do not come from the subterm $\mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}']$. Then we may strip the substitution $\mathtt{L}$ from $\mathtt{C}_1$ using Lem. 95, which means that we are either in case **A** or case **B**, and we are done.
2. **Otherwise.** Then some of the substitution nodes in $\mathtt{L}$ come from the subterm $\mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}']$. So we have that $\mathtt{C}_1$ is a substitution context and that $\mathtt{L} = \mathtt{C}_1[\mathtt{L}_1]$ for some substitution context $\mathtt{L}_1$. Note that $\mathtt{L}_1$ is non-empty since otherwise $\mathtt{L}$ would be subsumed in $\mathtt{C}_1$, which has already been considered in the previous case. Since $\mathtt{L}_1$ is non-empty we have that $\mathtt{L}_1 = \tilde{\mathtt{L}}[x\backslash v\mathtt{L}']$. So $\mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}'] = t\tilde{\mathtt{L}}[x\backslash v\mathtt{L}']$. Then we may strip the substitution $\tilde{\mathtt{L}}$ from $\mathtt{C}_2[x\backslash v\mathtt{L}']$ using Lem. 95. This gives us two possibilities, which correspond to cases **C** and **D** respectively.

$\square$

Lemma 97 (Answers do not have redexes or variables under non-answer contexts)**.** *Let* $v\mathtt{L}$ *be an answer. Then it cannot be the case that* $v\mathtt{L} = \mathcal{C}[\Delta]$ *where* $\mathcal{C} \in \mathcal{C}^{\cdot}_\vartheta$ *and* Δ *is a* $\mathtt{dB}$*,* $\mathtt{fix}$*,* $\mathtt{case}$*-redex or a variable.*

Proof. By the fact that non-answer evaluation contexts do not go below answers (Lem. 53) we have that $\mathtt{L} = \mathtt{L}_1\mathtt{L}_2$ and $\mathtt{C} = \mathtt{L}_2$. This means that $v\mathtt{L}_1$ is a $\mathtt{dB}$, $\mathtt{fix}$ or $\mathtt{case}$-redex or a variable, which is a contradiction. $\square$

Lemma 98 (Stripping a substitution from a term)**.** *Let $\mathcal{C}^h_\vartheta$ denote either $\mathcal{N}_\vartheta$, $\mathcal{S}_\vartheta$, $\mathcal{E}_\vartheta$, $\mathcal{K}_\vartheta$ or $\mathcal{L}_\vartheta$. If $t = t'\mathtt{L}$ and $t \in \mathcal{C}^h_\vartheta$, then $t' \in \mathcal{C}^h_{\hat\vartheta}$ where $\hat\vartheta \subseteq \mathsf{fz}^\vartheta(\mathtt{L})$.*

Proof. By induction on $\mathtt{L}$. If $\mathtt{L}$ is empty, it is immediate. If $\mathtt{L} = \mathtt{L}'[x\backslash u]$ there are two cases, depending on whether the rule nfSubG or the nfSub is applied to derive $t \in \mathbb{X}_\vartheta$.

1. **nfSubG,** $t = t'\mathtt{L}'[x\backslash u]$ **with** $x \notin \mathsf{ngv}(t'\mathtt{L}')$ **and** $t'\mathtt{L}' \in \mathbb{X}_\vartheta$**.** Then by *i.h.* $t'\mathtt{L}' \in \mathbb{X}_{\hat\vartheta}$ where $\hat\vartheta \subseteq \mathsf{fz}^\vartheta(\mathtt{L}') \subseteq \mathsf{fz}^\vartheta(\mathtt{L}'[x\backslash u])$.
2. **nfSub,** $t = t'\mathtt{L}'[x\backslash u]$ **with** $x \in \mathsf{ngv}(t'\mathtt{L}')$**,** $t'\mathtt{L}' \in \mathbb{X}_{\vartheta\cup\{x\}}$**, and** $u \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$**.** Then by *i.h.* $t'\mathtt{L}' \in \mathbb{X}_{\hat\vartheta}$ where $\hat\vartheta \subseteq \mathsf{fz}^{\vartheta\cup\{x\}}(\mathtt{L}') = \mathsf{fz}^\vartheta(\mathtt{L}'[x\backslash u])$.

□

Definition 99 (Reachable contexts)**.** *The set of reachable contexts is given by the following grammar:*

$$\begin{array}{rcl} R & ::= & \Box \mid R\,t \mid t\,R \mid \lambda x.R \mid R[x\backslash t] \mid R[\![x]\!][x\backslash R] \\ & & \mid \mathtt{fix}(x.R) \mid \mathtt{case}\ R\ \mathtt{of}\ \bar{b} \mid \mathtt{case}\ t\ \mathtt{of}\ (\mathbf{c}_1\bar{x}_1 \Rightarrow s_1)\ldots(\mathbf{c}_i\bar{x}_i \Rightarrow R)\ldots(\mathbf{c}_n\bar{x}_n \Rightarrow s_n) \end{array}$$

Lemma 100 (Reachable variables are variables below reachable contexts)**.** $\mathsf{ngv}(t) = \{x \mid \exists R.\ R$ *is a reachable context an* $R[\![x]\!]\}$

Proof. Straightforward by induction on t. The interesting case is when $t = s[x\backslash u]$. Note that in that case, by *i.h.*, we have that $x \in \mathsf{ngv}(s)$ if and only if s is of the form $R_1[\![x]\!]$. □

Lemma 101 (Evaluation contexts are reachable)**.** *Let $\mathtt{C} \in \mathcal{C}^h_\vartheta$ be a ϑ-evaluation context. Then $\mathtt{C}$ is reachable.*

Proof. Straightforward by induction on the derivation that $\mathtt{C} \in \mathcal{C}^h_\vartheta$. □

Lemma 102 (Replacing a variable in a reachable context yields a reachable context)**.** *Let $\widehat{\mathtt{C}}$ be a two-hole context such that $\widehat{\mathtt{C}}[\Box, y]$ is a reachable context and y is not bound by the context $\widehat{\mathtt{C}}[p, \Box]$ (for an arbitrary term p). Then for any term s the context $\widehat{\mathtt{C}}[\Box, s]$ is reachable.*

Proof. By induction on the derivation that $\widehat{\mathtt{C}}[\Box, y]$ is a reachable context:

1. **Empty,** $\widehat{\mathtt{C}}[\Box, y] = \Box$**.** Impossible.
2. **Under an abstraction,** $\widehat{\mathtt{C}}[\Box, y] = \lambda x.R$**.** Then $\widehat{\mathtt{C}}[\Box_1, \Box_2] = \lambda x.\widehat{\mathtt{C}}_1[\Box_1, \Box_2]$, so $\widehat{\mathtt{C}}[\Box, s] = \lambda x.\widehat{\mathtt{C}}_1[\Box, s]$ is reachable by *i.h.*.
3. **Left of an application,** $\widehat{\mathtt{C}}[\Box, y] = R\,t$**.** Two cases, depending on the position of the second hole of $\widehat{\mathtt{C}}$:
   - If the second hole of $\widehat{\mathtt{C}}$ is in t, *i.e.* $\widehat{\mathtt{C}}[\Box_1, \Box_2] = R[\Box_1]\,\mathtt{C}[\Box_2]$ with $t = \mathtt{C}[\![y]\!]$, then $\widehat{\mathtt{C}}[\Box, s] = R\,\mathtt{C}[s]$ is reachable.
   - If the second hole of $\widehat{\mathtt{C}}$ is in R, *i.e.* $\widehat{\mathtt{C}}[\Box_1, \Box_2] = \widehat{\mathtt{C}}_1[\Box_1, \Box_2]\,t$, then $\widehat{\mathtt{C}}[\Box, s] = \widehat{\mathtt{C}}_1[\Box, s]\,t$ is reachable by *i.h.*.
4. **Right of an application,** $\widehat{\mathtt{C}}[\Box, y] = t\,R$**.** Two cases, depending on the position of the second hole of $\widehat{\mathtt{C}}$:

- If the second hole of $\widehat{\mathtt{C}}$ is in t, *i.e.* $\widehat{\mathtt{C}}[\Box_1, \Box_2] = \mathtt{C}[\Box_2]\, R[\Box_1]$ with $t = \mathtt{C}[\![y]\!]$, then $\widehat{\mathtt{C}}[\Box, s] = \mathtt{C}[s]\, R$ is reachable.
- If the second hole of $\widehat{\mathtt{C}}$ is in R, *i.e.* $\widehat{\mathtt{C}}[\Box_1, \Box_2] = t\, \widehat{\mathtt{C}}_1[\Box_1, \Box_2]$, then $\widehat{\mathtt{C}}[\Box, s] = t\, \widehat{\mathtt{C}}_1[\Box, s]$ is reachable by *i.h.*.

5. **Left of a substitution,** $\widehat{\mathtt{C}}[\Box, y] = R[x\backslash t]$**.** Two cases, depending on the position of the second hole of $\widehat{\mathtt{C}}$:
   - If the second hole of $\widehat{\mathtt{C}}$ is in t, *i.e.* $\widehat{\mathtt{C}}[\Box_1, \Box_2] = R[\Box_1][x\backslash \mathtt{C}[\Box_2]]$ with $t = \mathtt{C}[\![y]\!]$, then $\widehat{\mathtt{C}}[\Box, s] = R[x\backslash \mathtt{C}[s]]$ is reachable.
   - If the second hole of $\widehat{\mathtt{C}}$ is in R, *i.e.* $\widehat{\mathtt{C}}[\Box_1, \Box_2] = \widehat{\mathtt{C}}_1[\Box_1, \Box_2][x\backslash t]$, then $\widehat{\mathtt{C}}[\Box, s] = \widehat{\mathtt{C}}_1[\Box, s][x\backslash t]$ is reachable by *i.h.*.
6. **Inside a substitution,** $\widehat{\mathtt{C}}[\Box, y] = R_1[\![x]\!][x\backslash R_2]$**.** Two cases, depending on the position of the second hole of $\widehat{\mathtt{C}}$:
   - If the second hole of $\widehat{\mathtt{C}}$ is in $R_1[\![x]\!]$, *i.e.* $\widehat{\mathtt{C}}[\Box_1, \Box_2] = \widehat{\mathtt{C}}_1[x, \Box_2][x\backslash R_2[\Box_1]]$ with $R_1 = \widehat{\mathtt{C}}_1[\Box, y]$, then by *i.h.* $\widehat{\mathtt{C}}_1[\Box, s]$ is reachable, so $\widehat{\mathtt{C}}[\Box, s] = \widehat{\mathtt{C}}_1[x, s][x\backslash R_2]$ is also reachable.
   - If the second hole of $\widehat{\mathtt{C}}$ is in R_2, *i.e.* $\widehat{\mathtt{C}}[\Box_1, \Box_2] = R_1[\![x]\!][x\backslash \widehat{\mathtt{C}}_1[\Box_1, \Box_2]]$, with $R_2 = \widehat{\mathtt{C}}_1[\Box, y]$, then by *i.h.* $\widehat{\mathtt{C}}_1[\Box, s]$ is reachable, so $\widehat{\mathtt{C}}[\Box, s] = R_1[\![x]\!][x\backslash \widehat{\mathtt{C}}_1[\Box, s]]$ is also reachable.
7. **Under a fix,** $\widehat{\mathtt{C}}[\Box, y] = \mathtt{fix}(x.R)$**.** Then $\widehat{\mathtt{C}}[\Box_1, \Box_2] = \mathtt{fix}(x.\widehat{\mathtt{C}}_1[\Box_1, \Box_2])$, so $\widehat{\mathtt{C}}[\Box, s] = \mathtt{fix}(x.\widehat{\mathtt{C}}_1[\Box, s])$ is reachable by *i.h.*.
8. **Under the target of a case,** $\widehat{\mathtt{C}}[\Box, y] = \mathtt{case}\ R\ \mathtt{of}\ \bar{b}$**.** Similar to the case **left of an application**.
9. **Under a branch of a case,** $\widehat{\mathtt{C}}[\Box, y] = \mathtt{case}\ t\ \mathtt{of}\ (\mathbf{c}_1 \bar{x}_1 \Rightarrow s_1) \ldots (\mathbf{c}_i \bar{x}_i \Rightarrow R) \ldots (\mathbf{c}_n \bar{x}_n \Rightarrow s_n)$**.** Similar to the case **right of an application**.

□

Lemma 103 (Preservation of reachable variables by internal steps when going to normal form)**.** *Let $t \to_{\mathtt{sh}}^{\neg\vartheta} s$ be a ϑ-internal step, such that $s \in \mathcal{N}_\vartheta$ is a strong ϑ-normal form. Then* $\mathsf{ngv}(t) \subseteq \mathsf{ngv}(s)$.

Proof. Let $\mathtt{r} : t \to_{\mathtt{sh}}^{\neg\vartheta} s$ be the internal step. The proof goes by induction on t.

1. **Variable,** $t = x$**.** Impossible.
2. **Abstraction,** $t = \lambda x.t'$**.** Let $\mathtt{r}_1 : t' \to_{\mathtt{sh}\backslash\mathtt{gc}} s'$ be the step isomorphic to $\mathtt{r}$ but going under the lambda. Note that $\mathtt{r}_1$ cannot be $(\vartheta \cup \{x\})$-external, for otherwise $\mathtt{r}$ would be ϑ-external. Let $y \in \mathsf{ngv}(\lambda x.t') = \mathsf{ngv}(t') \setminus \{x\}$. Then by *i.h.* $y \in \mathsf{ngv}(s') \setminus \{x\} = \mathsf{ngv}(\lambda x.s')$.
3. **Application,** $t = t_1\, t_2$**.** Note that $\mathtt{r}$ cannot be a step at the root, since it would be a $\mathtt{dB}$ step, and it would be external. Hence there are two cases, depending on whether the step $\mathtt{r}$ is internal to t_1 or internal to t_2:
   (a) **If $\mathtt{r}$ is internal to t_1.** Let $\mathtt{r}_1 : t_1 \to_{\mathtt{sh}\backslash\mathtt{gc}} s_1$ be the step isomorphic to $\mathtt{r}$ but going under the context $\Box\, t_2$. Then $s = s_1\, t_2$. Note that $\mathtt{r}_1$ cannot be ϑ-external, for otherwise $\mathtt{r}$ would be ϑ-external. So:
   $$\mathsf{ngv}(t_1\, t_2) = \mathsf{ngv}(t_1) \cup \mathsf{ngv}(t_2) \subseteq^{i.h.} \mathsf{ngv}(s_1) \cup \mathsf{ngv}(t_2) = \mathsf{ngv}(s_1\, t_2)$$

(b) **If $\mathtt{r}$ is internal to t_2.** Let $\mathtt{r}_1 : t_2 \to_{\mathtt{sh\backslash gc}} s_2$ be the step isomorphic to $\mathtt{r}$ but going under the context $t_1\,\Box$. Then $s = t_1\,s_2$. Recall that by hypothesis $s \in \mathcal{N}_\vartheta$ is a normal form, so t_1 must be a strong ϑ-structure, *i.e.* $t_1 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$. The step $\mathtt{r}_1$ cannot be ϑ-external, for otherwise $\mathtt{r}$ would be ϑ-external (note that this depends on the fact that t_1 is a structure). So:

$$\mathsf{ngv}(t_1\,t_2) = \mathsf{ngv}(t_1) \cup \mathsf{ngv}(t_2) \subseteq^{i.h.} \mathsf{ngv}(t_1) \cup \mathsf{ngv}(s_2) = \mathsf{ngv}(t_1\,s_2)$$

4. **Substitution, $t = t_1[x\backslash t_2]$.** We consider three cases, depending on whether (1) the step $\mathtt{r}$ is at the root of t, (2) $\mathtt{r}$ is internal to t_1, (3) $\mathtt{r}$ is internal to t_2.

(a) **If $\mathtt{r}$ is at the root of t.** Then $\mathtt{r}$ is a $\mathtt{lsv}$ step, which means that $t_1 = \mathtt{C}[\![x]\!]$ and $t_2 = v\mathtt{L}$ in such a way that:

$$\mathtt{r} : t = \mathtt{C}[\![x]\!][x\backslash v\mathtt{L}] \to_{\mathtt{sh}}^{\neg\vartheta} \mathtt{C}[v][x\backslash v]\mathtt{L} = s$$

Since $s = \mathtt{C}[v][x\backslash v]\mathtt{L} \in \mathcal{N}_\vartheta$ we may strip the substitution context $[x\backslash v]\mathtt{L}$ (by Lem. 98) to obtain that $\mathtt{C}[v] \in \mathcal{N}_{\hat{\vartheta}}$ where $\hat{\vartheta} \subseteq \mathsf{fz}^\vartheta([x\backslash v]\mathtt{L}) = \mathsf{fz}^\vartheta(\mathtt{L})$. We consider two cases, depending on whether $\mathtt{C}[\![x]\!]$ is a normal form in $\mathcal{N}_{\hat{\vartheta}}$:

i. **If $\mathtt{C}[\![x]\!] \in \mathcal{N}_{\hat{\vartheta}}$.** We consider two further subcases, depending on whether x is a reachable variable in $\mathtt{C}[\![x]\!]$:

A. **If $x \in \mathsf{ngv}(\mathtt{C}[\![x]\!])$.** Recall that $\hat{\vartheta} \subseteq \vartheta \cup \mathrm{dom}\,\mathtt{L}$. Moreover, observe that $\mathtt{C}[\![x]\!]$ is outside the scope of $\mathtt{L}$ in the original term $\mathtt{C}[\![x]\!][x\backslash v\mathtt{L}]$, so by Barendregt's convention we may suppose that variables in $\mathrm{dom}\,\mathtt{L}$ do not occur in $\mathtt{C}[\![x]\!]$. In particular, variables in $\mathrm{dom}\,\mathtt{L}$ are not reachable in $\mathtt{C}[\![x]\!]$, so by the fact that unreachable variables are not required in "ϑ" (Lem. 62) we have $\mathtt{C}[\![x]\!] \in \mathcal{N}_\vartheta$.
Since $x \in \mathsf{ngv}(\mathtt{C}[\![x]\!])$ and $\mathtt{C}[\![x]\!]$ is a normal form in $\mathcal{N}_\vartheta$, by the fact that reachable variables in normal forms are below evaluation contexts (Lem. 82), we know that there exists an evaluation context $\mathtt{C} \in \mathcal{C}^h_\vartheta$ such that $\mathtt{C}[\![x]\!] = \mathtt{C}[x]$. There are two subcases, depending on whether $\mathtt{C} = \mathtt{C}$ or $\mathtt{C} \neq \mathtt{C}$:

- **If $\mathtt{C} = \mathtt{C}$.** Then $\mathtt{C}[x\backslash v\mathtt{L}]$ is an evaluation context in $\mathcal{C}^h_\vartheta$, contradicting the fact that $\mathtt{r}$ is ϑ-internal.
- **If $\mathtt{C} \neq \mathtt{C}$.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\Box, x] = \mathtt{C} \qquad \widehat{\mathtt{C}}[x, \Box] = \mathtt{C}$$

And the step is of the form:

$$\mathtt{r} : t = \widehat{\mathtt{C}}[\underline{x}, x][x\backslash v\mathtt{L}] \to_{\mathtt{sh\backslash gc}} \widehat{\mathtt{C}}[\underline{x}, v][x\backslash v]\mathtt{L} = s$$

Note that the underlined occurrence of x is reachable on the left-hand side, so it is also reachable on the right-hand side.
More precisely, $\widehat{\mathtt{C}}[\Box, x] = \mathtt{C}$ is an evaluation context so by Lem. 101 it is also a reachable context. Recall that replacing a variable by an arbitrary term in a reachable context is still a reachable context (Lem. 102), so $\widehat{\mathtt{C}}[\Box, v]$ is also reachable. Moreover, since reachable variables coincide with variables below reachable contexts (Lem. 100) we have that $x \in \mathsf{ngv}(\widehat{\mathtt{C}}[x, v])$.
This contradicts the fact that s is a normal form, since to conclude that $\widehat{\mathtt{C}}[x, v][x\backslash v\mathtt{L}]$ is a normal form, given that $x \in \mathsf{ngv}(\widehat{\mathtt{C}}[x, v])$, we would require that x is bound to a structure, but it is bound to a value v.

B. **If $x \notin \mathsf{ngv}(\mathtt{C}[\![x]\!])$.** Let us show that $\mathsf{ngv}(t) \subseteq \mathsf{ngv}(s)$. Consider an arbitrary variable $y \in \mathsf{ngv}(t) = \mathsf{ngv}(\mathtt{C}[\![x]\!][x\backslash v\mathtt{L}])$, and let us show that $y \in \mathsf{ngv}(s)$. Since x is not reachable in $\mathtt{C}[\![x]\!]$, it must be the case that $y \in \mathsf{ngv}(\mathtt{C}[\![x]\!])$. Moreover, since $x \neq y$ and y is reachable in $\mathtt{C}[\![x]\!]$, by the fact that reachable variables are below reachable contexts (Lem. 100) there must exist a two-hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\Box, x] \text{ is reachable} \qquad \widehat{\mathtt{C}}[y, \Box] = \mathtt{C}$$

By replacing a variable in a reachable context (Lem. 102) we obtain that $\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}$ is also reachable. So $y \in \mathsf{ngv}(\widehat{\mathtt{C}}[y, v]) = \mathsf{ngv}(\mathtt{C}[v])$. Hence $y \in \mathsf{ngv}(\mathtt{C}[v][x\backslash v]\mathtt{L}) = \mathsf{ngv}(s)$, as required.

ii. **If $\mathtt{C}[\![x]\!] \notin \mathcal{N}_{\hat{\vartheta}}$.** Then by definition (Def. 83) $\mathtt{C}$ is a $\mathcal{N}_{\hat{\vartheta}}$-critical context. By Lem. 84 since $\mathtt{C}$ is $\mathbb{X}_{\hat{\vartheta}}$-critical, it is an evaluation context, $\mathtt{C} \in \mathcal{C}^h_{\hat{\vartheta}}$.

Recall that $\hat{\vartheta} \subseteq \vartheta \cup \mathtt{L}$. Moreover, the context $\mathtt{C}$ is outside the scope of $\mathtt{L}$ in the original term $\mathtt{C}[\![x]\!][x\backslash v\mathtt{L}]$, so by Barendregt's convention we may suppose that variables in $\mathrm{dom}\,\mathtt{L}$ do not occur in $\mathtt{C}$. In particular, variables in $\mathrm{dom}\,\mathtt{L}$ are not structural variables in $\mathtt{C}$, so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87), we may conclude that $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$.

Then also $\mathtt{C}[x\backslash v\mathtt{L}] \in \mathcal{C}^h_{\vartheta}$, contradicting the fact that the step $\mathtt{r}$ is ϑ-internal.

(b) **If $\mathtt{r}$ is internal to t_1.** Let $\mathtt{r}_1 : t_1 \to s_1$ be the step isomorphic to $\mathtt{r}$ but going under the context $[x\backslash t_2]$. Then $s = s_1[x\backslash t_2]$. Note that $\mathtt{r}_1$ cannot be ϑ-external, since then $\mathtt{r}$ would be ϑ-external. There are two cases, depending on whether x is reachable in t_1 or not:

i. **If $x \in \mathsf{ngv}(t_1)$.** Note that by *i.h.* $x \in \mathsf{ngv}(s_1)$. Then $\mathsf{ngv}(t) = \mathsf{ngv}(t_1) \cup \mathsf{ngv}(t_2) \subseteq^{i.h.} \mathsf{ngv}(s_1) \cup \mathsf{ngv}(t_2) = \mathsf{ngv}(s_1[x\backslash t_2]) = \mathsf{ngv}(s)$.

ii. **If $x \notin \mathsf{ngv}(t_1)$.** Then $\mathsf{ngv}(t) = \mathsf{ngv}(t_1) \subseteq^{i.h.} \mathsf{ngv}(s_1) \subseteq \mathsf{ngv}(s_1[x\backslash t_2]) = \mathsf{ngv}(s)$.

(c) **If $\mathtt{r}$ is internal to t_2.** Let $\mathtt{r}_1 : t_2 \to s_2$ be the step isomorphic to $\mathtt{r}$ but going under the context $t_1[x\backslash\Box]$. Then $s = t_1[x\backslash s_2]$. We consider two subcases, depending on whether x is reachable in t_1 or not:

i. **If $x \in \mathsf{ngv}(t_1)$.** We consider two subcases, depending on whether $\mathtt{r}_1$ is ϑ-external or ϑ-internal:

A. **If $\mathtt{r}$ is ϑ-external.** Since $t_1[x\backslash s_2]$ is a normal form, we know that $t_1 \in \mathcal{N}_{\vartheta\cup\{x\}}$. By the fact that reachable variables in normal forms are below evaluation contexts (Lem. 82) there must exist an evaluation context $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta}$ such that $t_1 = \mathtt{C}_1[\![x]\!]$. Moreover, since the step $\mathtt{r}_1$ is external, we have that $t_2 = \mathtt{C}_2[\![\Sigma]\!]$ where $\mathtt{C}_2 \in \mathcal{C}^h_{\vartheta}$ and Σ is the anchor of a redex. If we let Σ' denote its contractum, we have that the step $\mathtt{r}$ is of the form:

$$\mathtt{r} : t = \mathtt{C}_1[\![x]\!][x\backslash \mathtt{C}_2[\Sigma]] \to^{\neg\vartheta}_{\mathtt{sh}} \mathtt{C}_1[\![x]\!][x\backslash \mathtt{C}_2[\Sigma']] = s$$

Note that $\mathtt{C}_2$ cannot be a non-answer ϑ-evaluation context, since otherwise the step $\mathtt{r}$ would be ϑ-external.

Hence we have that $\mathtt{C}_2 \notin \mathcal{C}^{\cdot}_{\vartheta}$. Recall that evaluation contexts which are not non-answer evaluation contexts have the shape of an answer (Lem. 45). In particular, the subterm $\mathtt{C}_2[\Sigma']$ is an answer $(\lambda y.r)\mathtt{L}$. This contradicts the hypothesis that

$s = \mathtt{C}_1[\![x]\!][x\backslash(\lambda y.r)\mathtt{L}]$ is in normal form, since x is below an evaluation context and bound to an answer.

B. **If $\mathtt{r}$ is ϑ-internal.** Then $\mathsf{ngv}(t) = \mathsf{ngv}(t_1) \cup \mathsf{ngv}(t_2) \subseteq^{i.h.} \mathsf{ngv}(s_1) \cup \mathsf{ngv}(t_2) = \mathsf{ngv}(s)$ as required.

ii. **If $x \notin \mathsf{ngv}(t_1)$.** Then $\mathsf{ngv}(t) = \mathsf{ngv}(t_1) = \mathsf{ngv}(s)$ and we are done.

5. **Fix, $t = \mathtt{fix}(x.t_1)$.** Impossible since the reduction step would have to be inside t_1 and then s would not be in $\mathcal{N}_\vartheta$.

6. **Case, $t = \mathtt{case}\ t_0\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n$.** The $\mathtt{r}$ step cannot be at the root since it would be external. So it must be inside one of the t_i, with $i \in 0..n$. We reason as in the previous cases using the *i.h.*.

□

Lemma 104 (Backwards preservation of strong normal forms by internal steps)**.** *Let $t_0 \to_{\mathtt{sh}}^{\neg\vartheta} t$ be an internal step with $t \in \mathbb{X}_\vartheta$ where $\mathbb{X}_\vartheta$ stands for either $\mathcal{N}_\vartheta$, $\mathcal{S}_\vartheta$, $\mathcal{E}_\vartheta$, $\mathcal{K}_\vartheta$ or $\mathcal{L}_\vartheta$. Then $t_0 \in \mathbb{X}_\vartheta$.*

Proof. By induction on the derivation that $t \in \mathcal{C}^h_\vartheta$.

1. **lNfLam, $t = \lambda x.t' \in \mathcal{N}_\vartheta$ with $t' \in \mathcal{N}_{\vartheta\cup\{x\}}$.** Note that the step $\mathtt{r}$ cannot be a $\mathtt{dB}$, $\mathtt{lsv}$ or $\mathtt{fix}$ step at the root of t_0, since the right-hand side of these rules is a substitution. Then t_0 must be of the form $\lambda x.t'_0$ and $\mathtt{r}$ must be internal to t'_0. Let $\mathtt{r}_1 : t'_0 \to_{\mathtt{sh\backslash gc}} N^{\vartheta\cup\{x\}}$ be the step isomorphic to $\mathtt{r}$ but going under the lambda. Then by *i.h.* $t'_0 \in \mathcal{N}_\vartheta$, so indeed $t_0 = \lambda x.t'_0 \in \mathcal{N}_\vartheta$.

   If the step $\mathtt{r}$ is at the root of t_0, then it must be a $\mathtt{case}$ step of the form $\mathtt{case}\ \mathbf{c}_j\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j \Rightarrow \lambda x.t', \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n \to \lambda x.t'$. But this is an external step.

2. **n-var, $t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ with $x \in \vartheta$.** This case is impossible since, regardless of whether the step $\mathtt{r}$ is a $\mathtt{dB}$ or a $\mathtt{lsv}$ step, the right-hand side of $\mathtt{r}$ always contains at least one substitution.

3. **nfStruct.** Recall that the rule NFSTRUCT allows us to conclude that $t \in \mathcal{N}_\vartheta$ from the premise that $t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$. Then by *i.h.* $t_0 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$, and applying the rule NFSTRUCT we conclude $t_0 \in \mathcal{N}_\vartheta$.

4. **n-app, $t = u\,N^\vartheta \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ with $u \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ and $N^\vartheta \in \mathcal{N}_\vartheta$.** Note that the step $\mathtt{r}$ cannot be at the root of t_0, since the right-hand side of both $\mathtt{dB}$ and $\mathtt{lsv}$ steps is a substitution, rather than an application.

   So t_0 is an application $t_1\,t_2$, and we consider two cases depending on whether the step $\mathtt{r}$ is internal to t_1 or internal to t_2:

   (a) **If $\mathtt{r}$ is to the left of $t_0 = t_1\,N^\vartheta$.** Let $\mathtt{r}_1 : t_1 \to_{\mathtt{sh\backslash gc}} u$ be the step isomorphic to $\mathtt{r}$ but going under the context $\Box\,N^\vartheta$. Note that $\mathtt{r}_1$ cannot be ϑ-external, since this would imply that $\mathtt{r}$ is ϑ-external. So $\mathtt{r}_1$ is ϑ-internal and by *i.h.* we have that $t_1 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$. Hence $t_0 = t_1\,N^\vartheta \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$, as required.

   (b) **If $\mathtt{r}$ is to the right of $t_0 = u\,t_2$.** Let $\mathtt{r}_1 : t_2 \to_{\mathtt{sh\backslash gc}} N^\vartheta$ be the step isomorphic to $\mathtt{r}$ but going under the context $u\,\Box$. Note that $\mathtt{r}_1$ cannot be ϑ-external, since this would imply that $\mathtt{r}$ is ϑ-external. So $\mathtt{r}_1$ is ϑ-internal and by *i.h.* we have that $t_2 \in \mathcal{N}_\vartheta$. Hence $t_0 = M^\vartheta\,t_2 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$, as required.

5. **nfSubG,** $t = s[x\backslash u] \in \mathbb{X}_\vartheta$ **with** $x \notin \mathsf{ngv}(s)$ **and** $s \in \mathbb{X}_\vartheta$**.** We consider three cases, depending on whether (1) $\mathtt{r}$ is a step at the root of t_0, (2) t_0 is a substitution $s_0[x\backslash u_0]$ and $\mathtt{r}$ is internal to t_1, (3) t_0 is a substitution $s_0[x\backslash u_0]$ and $\mathtt{r}$ is internal to t_2.

   (a) **If $\mathtt{r}$ is at the root.** Note that $\mathtt{r}$ cannot be a $\mathtt{dB}$ step since it would be external, it must be a $\mathtt{lsv}$ step:
   $$\mathtt{r} : t_0 = \mathtt{C}\llbracket y\rrbracket[y\backslash v\mathtt{L}] \to_{\mathtt{sh}}^{\neg\vartheta} \mathtt{C}[v][y\backslash v]\mathtt{L} = s[x\backslash u]$$
   So s is of the form $s = s'\mathtt{L}_1$ with $\mathtt{L}_1[x\backslash u] = [y\backslash v]\mathtt{L}$ and $\mathtt{C}[v] = s'$. Note that since $s = s'\mathtt{L}_1 \in \mathbb{X}_\vartheta$ by Lem. 98 we must have $s' \in \mathbb{X}_{\hat\vartheta}$ where $\hat\vartheta \subseteq \mathsf{fz}^\vartheta(\mathtt{L}_1[x\backslash u]) = \mathsf{fz}^\vartheta([y\backslash v]\mathtt{L}) = \mathsf{fz}^\vartheta(\mathtt{L})$.
   We consider two subcases, depending on whether $\mathtt{C}\llbracket y\rrbracket \in \mathbb{X}_{\hat\vartheta}$.

      i. **If $\mathtt{C}\llbracket y\rrbracket \in \mathbb{X}_{\hat\vartheta}$.** Note that $\hat\vartheta \subseteq \mathsf{fz}^\vartheta(\mathtt{L}) \subseteq \vartheta \cup \mathrm{dom}\,\mathtt{L}$. and variables in $\mathrm{dom}\,\mathtt{L}$ do not occur in $\mathtt{C}\llbracket y\rrbracket$, since $\mathtt{C}\llbracket y\rrbracket$ is outside the scope of $\mathtt{L}$ in the original term $t_0 = \mathtt{C}\llbracket y\rrbracket[y\backslash v\mathtt{L}]$. In particular, variables in $\mathrm{dom}\,\mathtt{L}$ are not reachable variables in $\mathtt{C}\llbracket y\rrbracket$, so by repeatedly applying the fact that unreachable variables are not required in "ϑ" (Lem. 62), we have that $\mathtt{C}\llbracket y\rrbracket \in \mathbb{X}_\vartheta$.
      Let us consider two further subcases, depending on whether y is a reachable variable in $\mathtt{C}\llbracket y\rrbracket$:

         A. **If $y \in \mathsf{ngv}(\mathtt{C}\llbracket y\rrbracket)$.** Recall that reachable variables in a normal form are below evaluation contexts (Lem. 82). Then since $\mathtt{C}\llbracket y\rrbracket$ is a normal form in $\mathcal{C}_\vartheta^h$ and $y \in \mathsf{ngv}(\mathtt{C}\llbracket y\rrbracket)$, we know that $\mathtt{C}\llbracket y\rrbracket$ may be written as $\mathtt{C}\llbracket y\rrbracket$, where $\mathtt{C} \in \mathcal{C}_\vartheta^h$.
         We consider two cases, depending on whether $\mathtt{C}$ and $\mathtt{C}$ are the same context or distinct contexts:

         - **If $\mathtt{C} = \mathtt{C}$.** Then $\mathtt{C} \in \mathcal{C}_\vartheta^h$, hence $\mathtt{C}[y\backslash v\mathtt{L}] \in \mathcal{C}_\vartheta^h$. This contradicts the hypothesis that the step $\mathtt{r}$ is ϑ-internal.
         - **If $\mathtt{C} \neq \mathtt{C}$.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:
           $$\widehat{\mathtt{C}}[\Box, y] = \mathtt{C} \qquad \widehat{\mathtt{C}}[y, \Box] = \mathtt{C}$$
           And the step $\mathtt{r}$ is of the form:
           $$\mathtt{r} : \widehat{\mathtt{C}}[\underline{y}, y][y\backslash v\mathtt{L}] \to_{\mathtt{sh}\backslash\mathtt{gc}} \widehat{\mathtt{C}}[\underline{y}, v][y\backslash v]\mathtt{L} = t$$
           Note that the underlined occurrence of y is reachable on the left-hand side, so it is also reachable on the right-hand side.
           More precisely, $\widehat{\mathtt{C}}[\Box, y] = \mathtt{C}$ is an evaluation context so by Lem. 101 it is also a reachable context. Recall that replacing a variable by an arbitrary term in a reachable context is still a reachable context (Lem. 102), so $\widehat{\mathtt{C}}[\Box, v]$ is also reachable. Moreover, since reachable variables coincide with variables below reachable contexts (Lem. 100) we have that $y \in \mathsf{ngv}(\widehat{\mathtt{C}}[y, v])$.
           This contradicts the fact that t is a normal form, since to conclude that $\widehat{\mathtt{C}}[y, v][y\backslash v]$ is a normal form, given that $y \in \mathsf{ngv}(\widehat{\mathtt{C}}[y, v])$ we would require that y is bound to a structure, but it is bound to a value v.

         B. **If $y \notin \mathsf{ngv}(\mathtt{C}\llbracket y\rrbracket)$.** Then we are done, as $\mathtt{C}\llbracket y\rrbracket \in \mathbb{X}_\vartheta$, so by applying the nfSubG rule we obtain that $\mathtt{C}\llbracket y\rrbracket[y\backslash v\mathtt{L}] \in \mathbb{X}_\vartheta$, as wanted.

ii. **If $\mathtt{C}[\![y]\!] \notin \mathbb{X}_{\hat{\vartheta}}$.** Note that $\mathtt{C}[v] = s' \in \mathbb{X}_{\hat{\vartheta}}$. So by definition (Def. 83) $\mathtt{C}$ is a $\mathbb{X}_{\hat{\vartheta}}$-critical context. By Lem. 84 since $\mathtt{C}$ is $\mathbb{X}_{\hat{\vartheta}}$-critical, it is an evaluation context, $\mathtt{C} \in \mathbb{X}_{\hat{\vartheta}}$. Moreover, note that $\hat{\vartheta} \subseteq \mathsf{fz}^{\vartheta}(\mathtt{L}) \subseteq \vartheta \cup \operatorname{dom}\mathtt{L}$. and variables in $\operatorname{dom}\mathtt{L}$ do not occur in $\mathtt{C}$, since $\mathtt{C}$ is outside the scope of $\mathtt{L}$ in the original term $t_0 = \mathtt{C}[\![y]\!][y\backslash v\mathtt{L}]$. In particular, variables in $\operatorname{dom}\mathtt{L}$ are not structural variables in $\mathtt{C}$, so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87), we may conclude that $\mathtt{C} \in \mathbb{X}_{\vartheta}$. Then the context $\mathtt{C}[y\backslash v\mathtt{L}]$ is an evaluation context in $\mathbb{X}_{\vartheta}$, contradicting the fact that the step $\mathtt{r}$ is ϑ-internal.

(b) **If $\mathtt{r}$ is to the left of $t_0 = s_0[x\backslash u]$.** Let $\mathtt{r}_1 : s_0 \to_{\mathtt{sh\backslash gc}} s$ be the step isomorphic to $\mathtt{r}$ but going under the context $\Box[x\backslash u]$. Note that $\mathtt{r}_1$ cannot be ϑ-external, since then $\mathtt{r}$ would be ϑ-external. So $\mathtt{r}_1$ is ϑ-internal and by *i.h.* we have that $s_0 \in \mathcal{C}^h_{\vartheta}$. Moreover, since reachable variables are preserved by internal steps (Lem. 103), by the contrapositive we have that $x \notin \mathsf{ngv}(s_0)$, hence $t_0 = s_0[x\backslash u] \in \mathcal{C}^h_{\vartheta}$ as required.

(c) **If $\mathtt{r}$ is to the right of $t_0 = s[x\backslash u_0]$.** Then by applying the rule NFSUBG it is immediate that $t_0 = s[x\backslash u_0] \in \mathcal{C}^h_{\vartheta}$

6. **nfSub,** $t = s[x\backslash M^{\vartheta}] \in \mathbb{X}_{\vartheta}$ **with** $x \in \mathsf{ngv}(s)$**,** $s \in \mathbb{X}_{\vartheta\cup\{x\}}$ **and** $M^{\vartheta} \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$**.** We consider three cases, depending on whether (1) $\mathtt{r}$ is a step at the root of t_0, (2) t_0 is a substitution $s_0[x\backslash u_0]$ and $\mathtt{r}$ is internal to t_1, (3) t_0 is a substitution $s_0[x\backslash u_0]$ and $\mathtt{r}$ is internal to t_2.

(a) **If $\mathtt{r}$ is at the root.** Note that $\mathtt{r}$ cannot be a $\mathtt{dB}$ step since it would be external, it must be a $\mathtt{lsv}$ step:

$$\mathtt{r} : t_0 = \mathtt{C}[\![y]\!][y\backslash v\mathtt{L}] \to^{\neg\vartheta}_{\mathtt{sh}} \mathtt{C}[v][y\backslash v]\mathtt{L} = s[x\backslash M^{\vartheta}]$$

So let us write s as of the form $s = s'\mathtt{L}_1$ in such a way that $\mathtt{L}_1[x\backslash M^{\vartheta}] = [y\backslash v]\mathtt{L}$. By Lem. 98 we have that $s' \in \mathbb{X}_{\hat{\vartheta}}$ where $\hat{\vartheta} \subseteq \mathsf{fz}^{\vartheta}(\mathtt{L}_1[x\backslash M^{\vartheta}]) = \mathsf{fz}^{\vartheta}([y\backslash v]\mathtt{L}) = \mathsf{fz}^{\vartheta}(\mathtt{L})$. Then the remainder of this case is analogous to case 5a of this lemma.

(b) **If $\mathtt{r}$ is to the left of $t_0 = s_0[x\backslash M^{\vartheta}]$.** Let $\mathtt{r}_1 : s_0 \to_{\mathtt{sh\backslash gc}} s$ be the step isomorphic to $\mathtt{r}$ but going under the context $\Box[x\backslash M^{\vartheta}]$. Note that $\mathtt{r}_1$ cannot be $(\vartheta \cup \{x\})$-external, since then $\mathtt{r}$ would be ϑ-external. So $\mathtt{r}_1$ is $(\vartheta \cup \{x\})$-internal, and since $s \in \mathbb{X}_{\vartheta\cup\{x\}}$ by *i.h.* we have that $s_0 \in \mathbb{X}_{\vartheta\cup\{x\}}$. We consider two further subcases, depending on whether x is reachable in s_0:

i. **If $x \in \mathsf{ngv}(s_0)$.** Then $s_0[x\backslash M^{\vartheta}] \in \mathbb{X}_{\vartheta}$ since $s_0 \in \mathbb{X}_{\vartheta\cup\{x\}}$, by the rule NFSUB.

ii. **If $x \notin \mathsf{ngv}(s_0)$.** Then since unreachable variables are not required in "ϑ" (Lem. 62), we have that $s_0 \in \mathbb{X}_{\vartheta}$. Hence $s_0[x\backslash M^{\vartheta}] \in \mathbb{X}_{\vartheta}$, by the rule NFSUBG.

(c) **If $\mathtt{r}$ is to the right of $t_0 = s[x\backslash u_0]$.** Let $\mathtt{r}_1 : u_0 \to_{\mathtt{sh\backslash gc}} M^{\vartheta}$ be the step isomorphic to $\mathtt{r}$ but going under the context $s[x\backslash\Box]$. We consider two cases, depending on whether $\mathtt{r}_1$ is ϑ-external or ϑ-internal:

i. **If $\mathtt{r}_1$ is ϑ-external.** First note that, since $x \in \mathsf{ngv}(s)$ and $s \in \mathbb{X}_{\vartheta\cup\{x\}}$, by the fact that reachable variables in normal forms are below evaluation contexts (Lem. 82) there must exist an evaluation context $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta}$ such that $s = \mathtt{C}_1[\![x]\!]$.
Moreover, since $\mathtt{r}_1$ is a ϑ-external step, the term u_0 can be written as $\mathtt{C}_2[\Sigma]$, where $\mathtt{C}_2$ is an evaluation context in $\mathcal{C}^h_{\vartheta}$ and Σ is the anchor of a redex. If we let Σ' denote the contractum of Σ, the step $\mathtt{r}$ is:

$$\mathtt{r} : \mathtt{C}_1[\![x]\!][x\backslash\mathtt{C}_2[\Sigma]] \to^{\neg\vartheta}_{\mathtt{sh}} \mathtt{C}_1[\![x]\!][x\backslash\mathtt{C}_2[\Sigma']] = s[x\backslash M^{\vartheta}] = t$$

Since we know that the step $\mathtt{r}$ is ϑ-internal, the context $\mathtt{C}_2$ cannot be a non-answer evaluation context, *i.e.* $\mathtt{C}_2 \notin \mathcal{C}^{\cdot}_{\vartheta}$. Recall that evaluation contexts which are not non-answer evaluation contexts have the shape of an answer (Lem. 45). This means that $\mathtt{C}_2[\Sigma'] = (\lambda y.r)\mathtt{L}$ is an answer. But we also had that $\mathtt{C}_2[\Sigma'] = M^{\vartheta}$, so it is both an answer and a structure, which is impossible.

ii. **If $\mathtt{r}_1$ is ϑ-internal.** Then by *i.h.* u_0 is a structure, *i.e.* $u_0 \in M^{\vartheta}$. Hence $s[x\backslash u_0] \in \mathbb{X}_{\vartheta}$, as required.

7. **eNfCase.** $t = \mathtt{case}\ s_0\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ **and** $t \in \mathcal{E}_{\vartheta}$ **and** $(s_i \in \mathcal{N}_{\vartheta\cup\bar{x}_i})_{i\in I}$ **.**

   Note that the step $\mathtt{r}$ cannot be a $\mathtt{dB}$, $\mathtt{lsv}$ or $\mathtt{fix}$ step at the root of t_0, since the right-hand side of these rules is a substitution. So the reduction must be in one of the s_i with $i \in 0..n$. We reason as in the above cases.

   If the step $\mathtt{r}$ is at the root of t_0, then it must be a $\mathtt{case}$ step of the form $\mathtt{case}\ \mathbf{c}_j\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j \Rightarrow t, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n \to t$. But this is an external step.

8. **eNfStrt.** $t = \mathtt{case}\ s_0\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ **and** $t \in \mathcal{K}_{\vartheta} \cup \mathcal{L}_{\vartheta} \cup \mathcal{S}_{\vartheta}$ **and** $t \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ **and** $(s_i \in \mathcal{N}_{\vartheta\cup\bar{x}_i})_{i\in I}$ **.**

   Note that the step $\mathtt{r}$ cannot be a $\mathtt{dB}$, $\mathtt{lsv}$ or $\mathtt{fix}$ step at the root of t_0, since the right-hand side of these rules is a substitution. So the reduction must be in one of the s_i with $i \in 0..n$. We reason as in the above cases.

   If the step $\mathtt{r}$ is at the root of t_0, then it must be a $\mathtt{case}$ step of the form $\mathtt{case}\ \mathbf{c}_j\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j \Rightarrow t, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n \to t$. But this is an external step.

□

Lemma 105 (Backwards preservation of needed variables by internal steps)**.** *Let $t_0 \to^{\neg\vartheta}_{\mathtt{sh}} \mathtt{C}[\![x]\!]$ be an internal step with $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$, and such that $\mathtt{C}$ does not bind x. Then there exists an evaluation context $\mathtt{C}_0 \in \mathcal{C}^h_{\vartheta}$ such that $t_0 = \mathtt{C}_0[\![x]\!]$.*

Proof. Let $\mathtt{r}$ be the name of the ϑ-internal step $\mathtt{r} : t_0 \to^{\neg\vartheta}_{\mathtt{sh}} \mathtt{C}[\![x]\!]$. The proof goes by induction on the derivation that $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$.

1. **EBox,** $\mathtt{C} = \Box$**.** Then $\mathtt{r} : t_0 \to^{\neg\vartheta}_{\mathtt{sh}} x$, which is impossible, as the right-hand side of both $\mathtt{dB}$, $\mathtt{lsv}$ and $\mathtt{fix}$ steps always have at least one substitution and a $\mathtt{case}$ reduction at the root would be external.

2. **EAppL,** $\mathtt{C} = \mathtt{C}_2\, t$ **with** $\mathtt{C}_2 \in \mathcal{C}^{\cdot}_{\vartheta}$**.** Note that $\mathtt{r}$ cannot be at the root of t_0: it cannot be a $\mathtt{dB}$ or $\mathtt{case}$ step at the root, since it would be external, and it cannot be a $\mathtt{lsv}$ or $\mathtt{fix}$ step at the root, since then the right-hand side would have a substitution at the root (but it is an application).

   So t_0 is an application and there are two cases, depending on whether $\mathtt{r}$ is internal to the left or to the right of t_0:

   (a) **The step $\mathtt{r}$ is internal to the left of $t_0 = t'_0\, t$.** Consider the step $\mathtt{r}_1 : t'_0 \to_{\mathtt{sh}\backslash\mathtt{gc}} \mathtt{C}_2[\![x]\!]$ isomorphic to $\mathtt{r}$ but going under the context $\Box\, t$. Note that $\mathtt{r}_1$ must be a ϑ-internal step, for otherwise $\mathtt{r}$ would be ϑ-external. By *i.h.* there exists a non-answer evaluation context $\mathtt{I}^{\vartheta}_0 \in \mathcal{C}^{\cdot}_{\vartheta}$ such that $t'_0 = \mathtt{I}^{\vartheta}_0[\![x]\!]$. So we conclude by taking $\mathtt{C}_0 := \mathtt{I}^{\vartheta}_0\, t$.

(b) **The step $\mathtt{r}$ is internal to the right of $t_0 = \mathtt{C}_2[\![x]\!]\,t_0'$.** Consider the step $\mathtt{r}_1 : t_0' \to_{\mathtt{sh\backslash gc}} t$ isomorphic to $\mathtt{r}$ but going under the context $\mathtt{C}_2[\![x]\!]\,\Box$. Then it is immediate to conclude by taking $\mathtt{C}_0 := \mathtt{C}_2\,t_0'$.

3. **ESubLNonStr, $\mathtt{C} = \mathtt{C}_1[y\backslash t]$ with $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$ and $t \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** We consider three cases, depending on whether (1) the internal step $\mathtt{r}$ is at the root of t_0, (2) t_0 is a substitution $t_0'[y\backslash r_0]$ and the step $\mathtt{r}$ is internal to t_0', (3) t_0 is a substitution $t_0'[y\backslash r_0]$ and the step $\mathtt{r}$ is internal to r_0.

(a) **The internal step $\mathtt{r}$ is at the root of t_0.** Note that $\mathtt{r}$ cannot be a $\mathtt{dB}$, $\mathtt{fix}$ or $\mathtt{case}$ step, since it would be external. So it is an $\mathtt{lsv}$ step of the form:

$$\mathtt{r} : t_0 = \mathtt{C}[\![z]\!][z\backslash v\mathtt{L}] \to^{\neg\vartheta}_{\mathtt{sh}} \mathtt{C}[v][z\backslash v]\mathtt{L} = \mathtt{C}_1[\![x]\!][y\backslash t] = t_1$$

Let $\mathtt{L}_1$ be the substitution context such that $\mathtt{L}_1[y\backslash t] = [z\backslash v]\mathtt{L}$, and using Lem. 95 let us strip the substitution $\mathtt{L}_1$ from $\mathtt{C}_1[\![x]\!]$. This gives us two possibilities, **A** and **B**:

i. **Case A.** Then:

$$\mathtt{C}_1 = \mathtt{C}_{11}\mathtt{L}_1 \qquad \mathtt{C}[v] = \mathtt{C}_{11}[\![x]\!]$$

where $\hat{\vartheta} = \mathsf{fz}^\vartheta(\mathtt{L}_1)$ and $\mathtt{C}_{11} \in \mathbb{X}_{\hat{\vartheta}}$.
We consider three further subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{11}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{11}$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}_{11} \qquad \widehat{\mathtt{C}}[x, \Box] = \mathtt{C}$$

By the fact that evaluation contexts are backwards-stable by substitutions (Lem. 89) there are two possibilities: the left and the right branch of the disjunction. Let us analyze each branch:

- **Left branch.** Then $\widehat{\mathtt{C}}[\Box, z] \in \mathbb{X}_{\hat{\vartheta}}$. Note that $\hat{\vartheta} = \mathsf{fz}^\vartheta([z\backslash v]\mathtt{L}) \subseteq \vartheta \cup \mathrm{dom}\,\mathtt{L}$, and that the subterm $\mathtt{C}[\![z]\!]$ is outside the scope of $\mathtt{L}$ on the left-hand side of the step $\mathtt{r}$, so variables in $\mathrm{dom}\,\mathtt{L}$ do not occur free in $\mathtt{C}[\![z]\!]$. In particular, variables in $\mathrm{dom}\,\mathtt{L}$ are not structural variables in $\widehat{\mathtt{C}}[\Box, z]$. By repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87), we conclude that $\widehat{\mathtt{C}}[\Box, z] \in \mathbb{X}_\vartheta$.
Hence $t_0 = \widehat{\mathtt{C}}[x, z][z\backslash v\mathtt{L}]$ and by taking $\mathtt{C}_0 := \widehat{\mathtt{C}}[\Box, z][z\backslash v\mathtt{L}] \in \mathbb{X}_\vartheta$ we conclude.
- **Right branch.** Then $\widehat{\mathtt{C}}[x, \Box] \in \mathbb{X}_{\hat{\vartheta}}$. Since $\hat{\vartheta} \subseteq \vartheta \cup \mathrm{dom}\,\mathtt{L}$, by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87), we have that $\widehat{\mathtt{C}}[x, \Box] \in \mathbb{X}_\vartheta$. This means that the step:

$$\mathtt{r} : \widehat{\mathtt{C}}[x, z][z\backslash v\mathtt{L}] \to_{\mathtt{sh\backslash gc}} \widehat{\mathtt{C}}[x, v][z\backslash v]\mathtt{L}$$

is ϑ-external, contradicting the hypothesis that it is internal.

B. **The context $\mathtt{C}$ is a prefix of the context $\mathtt{C}_{11}$.** By the decomposition of evaluation contexts lemma (Lem. 50) the context $\mathtt{C}$ must be an evaluation context in $\mathbb{X}_{\hat{\vartheta}}$. Since $\hat{\vartheta} \subseteq \vartheta \cup \mathrm{dom}\,\mathtt{L}$, by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87), we have that $\mathtt{C} \in \mathbb{X}_\vartheta$. This contradicts the fact that $\mathtt{r}$ is ϑ-internal.

C. **The context $\mathtt{C}_{11}$ is a prefix of the context $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{11}[\mathtt{C}_1]$, so $\mathtt{C}_1[v] = x$, which is impossible.

ii. **Case B.** Then:

$$\mathtt{C}_1 = \mathtt{C}_{11}[\![w]\!]\mathscr{L}\{\Box\} \qquad \mathtt{C}[v] = \mathtt{C}_{11}[\![w]\!] \qquad \mathtt{L}_1 = \Box\mathscr{L}\{x\}$$

where $\hat{\vartheta} = \mathsf{fz}^{\vartheta}(\mathtt{L}_1)$, the evaluation context $\mathtt{C}_{11}$ is in $\mathbb{X}_{\hat{\vartheta}}$, and $\mathscr{L}$ is a (ϑ, w)-chain context.

We consider three further subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{11}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{11}$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}_{11} \qquad \widehat{\mathtt{C}}[w, \Box] = \mathtt{C}$$

and the internal step $\mathtt{r}$ is of the form:

$$\mathtt{r} : t_0 = \widehat{\mathtt{C}}[w, z][z\backslash v\mathtt{L}] \rightarrow_{\mathtt{sh}}^{\neg\vartheta} \widehat{\mathtt{C}}[w, v][z\backslash v]\mathtt{L} = t_1$$

Note that w is bound by $[z\backslash v]\mathtt{L} = \Box\mathscr{L}\{x\}[y\backslash t]$ on the right-hand side of the step $\mathtt{r}$ since $\mathscr{L}$ is a (ϑ, w)-chain context. So w must be bound by $[z\backslash v\mathtt{L}]$ on the left-hand side of the step $\mathtt{r}$, for otherwise it would be free, and free variables cannot become bound by reduction.

Hence it must be the case that $w = z$. Note that w is bound to a term of the form $\mathtt{I}_1^{\vartheta_1}[\![w_1]\!]$ on the right-hand side of the step $\mathtt{r}$, and we have just argued that $w = z$, so $\mathtt{I}_1^{\vartheta_1}[\![w_1]\!] = v$. This is impossible since answers do not have variables below non-answer evaluation contexts (Lem. 97).

B. **The context $\mathtt{C}$ is a prefix of the context $\mathtt{C}_{11}$.** Then by the decomposition of evaluation contexts lemma (Lem. 50) the context $\mathtt{C}$ must be an evaluation context in $\mathbb{X}_{\hat{\vartheta}}$. Since $\hat{\vartheta} \subseteq \vartheta \cup \operatorname{dom}\mathtt{L}$, by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87), we obtain that $\mathtt{C} \in \mathbb{X}_{\vartheta}$. This contradicts the fact that $\mathtt{r}$ is ϑ-internal.

C. **The context $\mathtt{C}_{11}$ is a prefix of the context $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{11}[\mathtt{C}_1]$, so $\mathtt{C}_1[v] = w$, which is impossible.

(b) **The internal step $\mathtt{r}$ is to the left of $t_0 = t_0'[y\backslash t]$.** Let $\mathtt{r}_1 : t_0' \rightarrow_{\mathtt{sh}\backslash\mathtt{gc}} \mathtt{C}_1[\![x]\!]$ be the step isomorphic to $\mathtt{r}$ but going under the context $[y\backslash t]$. Then by *i.h.* there is an evaluation context $\mathtt{C}_{10} \in \mathbb{X}_{\vartheta}$ such that $t_0' = \mathtt{C}_{10}[\![x]\!]$. By taking $\mathtt{C}_0 := \mathtt{C}_{10}[y\backslash t] \in \mathbb{X}_{\vartheta}$ we conclude that $t_0 = \mathtt{C}_{10}[\![x]\!][y\backslash t]$, as required.

(c) **The internal step $\mathtt{r}$ is to the right of $t_0 = \mathtt{C}_1[\![x]\!][y\backslash t_0']$.** By taking $\mathtt{C}_0 := \mathtt{C}_{10}[y\backslash t_0'] \in \mathbb{X}_{\vartheta}$ we conclude that $t_0 = \mathtt{C}_{10}[\![x]\!][y\backslash t]$, as required.

4. **ESubLStr, $\mathtt{C} = \mathtt{C}_1[y\backslash u]$ with $\mathtt{C} \in \mathcal{C}^h_{\vartheta\cup\{y\}}$ and $u \in \mathcal{S}_{\vartheta} \cup \mathcal{E}_{\vartheta}$.** We consider three cases, depending on whether (1) the internal step $\mathtt{r}$ is at the root of t_0, (2) t_0 is a substitution $t_0'[y\backslash r_0]$ and the step $\mathtt{r}$ is internal to t_0', (3) t_0 is a substitution $t_0'[y\backslash r_0]$ and the step $\mathtt{r}$ is internal to r_0.

(a) **The internal step $\mathtt{r}$ is at the root of t_0.** Note that $\mathtt{r}$ cannot be a $\mathtt{dB}$ step, since it would be external. So $\mathtt{r}$ is an $\mathtt{lsv}$ step of the form:

$$\mathtt{r} : t_0 = \mathtt{C}[\![z]\!][z\backslash v\mathtt{L}] \rightarrow_{\mathtt{sh}}^{\neg\vartheta} \mathtt{C}[\![v]\!][z\backslash v]\mathtt{L} = \mathtt{C}_1[\![x]\!][y\backslash u] = t_1$$

Let $\mathtt{L}_1$ be the substitution context such that $[z\backslash v]\mathtt{L} = \mathtt{L}_1[y\backslash u]$, and using Lem. 95 let us strip the substitution $\mathtt{L}_1$ from $\mathtt{C}_1[\![x]\!]$. This gives us two possibilities, **A** and **B**:

i. **Case A.** Then:

$$\mathtt{C}_1 = \mathtt{C}_{11}\mathtt{L}_1 \qquad \mathtt{C}[v] = \mathtt{C}[\![x]\!]$$

where $\hat{\vartheta} \cup \{y\} = \mathsf{fz}^{\vartheta\cup\{y\}}(\mathtt{L}_1)$ and $\mathtt{C}_{11} \in \mathbb{X}_{\hat{\vartheta}\cup\{y\}}$. We consider three further subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{11}$:

A. **The hole of $\mathtt{C}$ and $\mathtt{C}_{11}$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\square, v] = \mathtt{C}_{11} \qquad \widehat{\mathtt{C}}[x, \square] = \mathtt{C}$$

By the fact that evaluation contexts are backwards-stable by substitutions (Lem. 89) there are two possibilities: the left and the right branch of the disjunction. Let us analyze each branch:

- **Left branch.** Then $\widehat{\mathtt{C}}[\square, z] \in \mathbb{X}_{\hat{\vartheta}\cup\{y\}}$. First note that $\hat{\vartheta} \subseteq \vartheta \cup \mathrm{dom}\,\mathtt{L}$, so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\widehat{\mathtt{C}}[\square, z] \in \mathbb{X}_{\vartheta\cup\{y\}}$.
Moreover, note that y is bound by the substitution context $\mathtt{L}_1[y\backslash u] = [z\backslash v]\mathtt{L}$, so either $y = z$ or $y \in \mathrm{dom}\,\mathtt{L}$. Observe that y cannot be equal to z, since y is bound to u and z is bound to v, and strong structures cannot be values, so it must be the case that $y \in \mathrm{dom}\,\mathtt{L}$.
Given that $y \in \mathrm{dom}\,\mathtt{L}$, the variable y cannot occur free in the subterm $\mathtt{C}[\![z]\!] = \widehat{\mathtt{C}}[x, z]$ on the left-hand side of the step $\mathtt{r}$, since this subterm is outside the scope of $\mathtt{L}$. In particular, y does not occur as a structural variable in the context $\widehat{\mathtt{C}}[\square, z]$. So applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we obtain that $\widehat{\mathtt{C}}[\square, z] \in \mathbb{X}_{\vartheta}$.
Then it is immediate to conclude, since by taking $\mathtt{C}_0 := \widehat{\mathtt{C}}[\square, z][z\backslash v\mathtt{L}] \in \mathbb{X}_{\vartheta}$, we have that $t_0 = \widehat{\mathtt{C}}[x, z][z\backslash v\mathtt{L}]$, as required.
- **Right branch.** Then $\widehat{\mathtt{C}}[x, \square] \in \mathbb{X}_{\hat{\vartheta}\cup\{y\}}$. By the same argument as in the previous left branch, we have that $\widehat{\mathtt{C}}[x, \square] \in \mathbb{X}_{\vartheta}$. Then, adding an arbitrary substitution (Lem. 80) we have that $\widehat{\mathtt{C}}[x, \square][y\backslash u] \in \mathbb{X}_{\vartheta}$, which implies that the step $\mathtt{r}$ is ϑ-external, contradicting the hypothesis.

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{11}$.** Then by the decomposition of evaluation contexts lemma (Lem. 50) the context $\mathtt{C}$ must be an evaluation context in $\mathbb{X}_{\hat{\vartheta}\cup\{y\}}$.
Note that $\hat{\vartheta} \subseteq \vartheta \cup \mathrm{dom}\,\mathtt{L}$, so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C} \in \mathbb{X}_{\vartheta\cup\{y\}}$.
Moreover, y is bound by $\mathtt{L}_1[y\backslash u] = [z\backslash v]\mathtt{L}$, and $y \neq z$, since y is bound to u and z is bound to v. Hence y cannot occur free in the subterm $\mathtt{C}[\![z]\!]$ on the left-hand side of the step $\mathtt{r}$. In particular, y does not occur as a structural variable in $\mathtt{C}$. So applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we obtain that $\mathtt{C} \in \mathbb{X}_{\vartheta}$.

Note that then $\mathtt{C}[z\backslash v\mathtt{L}] \in \mathbb{X}_{\vartheta}$, which means that the step $\mathtt{r}$ is ϑ-external, contradicting the hypothesis that it is ϑ-internal.

C. **The context $\mathtt{C}_{11}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{11}[\mathtt{C}_1]$, so $\mathtt{C}_1[v] = x$, which is impossible.

ii. **Case B.** Then:

$$\mathtt{C}_1 = \mathtt{C}_{11}[\![w]\!]\mathscr{L}\{\Box\} \qquad \mathtt{C}[v] = \mathtt{C}_{11}[\![w]\!] \qquad \mathtt{L}_1 = \Box\mathscr{L}\{x\}$$

where $\hat{\vartheta} \cup \{y\} = \mathtt{fz}^{\vartheta\cup\{y\}}(\mathtt{L}_1)$, the evaluation context $\mathtt{C}$ is in $\mathbb{X}_{\hat{\vartheta}\cup\{y\}}$, and $\mathscr{L}$ is a (ϑ, w)-chain context.

We consider three further subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{11}$:

A. **The hole of $\mathtt{C}$ and $\mathtt{C}_{11}$ are disjoint.** Then there is a two hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}_{11} \qquad \widehat{\mathtt{C}}[w, \Box] = \mathtt{C}$$

The step $\mathtt{r}$ is of the form:

$$\mathtt{r} : \widehat{\mathtt{C}}[w, z][z\backslash v\mathtt{L}] \rightarrow_{\mathtt{sh}}^{\neg\vartheta} \widehat{\mathtt{C}}[w, v][z\backslash v]\mathtt{L}$$

Note that w is bound by $\Box\mathscr{L}\{x\}[y\backslash u] = [z\backslash v]\mathtt{L}$, since $\mathscr{L}$ is a (ϑ, w)-chain context. Hence it must be the case that $w = z$, for otherwise, if it were the case that $w \in \operatorname{dom}\mathtt{L}$, w would occur free on the left-hand side of the step $\mathtt{r}$, since it occurs outside the scope of $\mathtt{L}$. This is impossible since free variables cannot become bound after a reduction step.

Note that w must be bound to a term of the form $\mathtt{I}_1^{\vartheta_1}[\![w_1]\!]$ and, since we have just argued that $w = z$, we have that $\mathtt{I}_1^{\vartheta_1}[\![w_1]\!] = v$. This is impossible since answers do not have variables below non-answer evaluation contexts (Lem. 97).

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{11}$.** Then by the decomposition of evaluation contexts lemma (Lem. 50) the context $\mathtt{C}$ must be an evaluation context in $\mathbb{X}_{\hat{\vartheta}\cup\{y\}}$.

Note that $\hat{\vartheta} \subseteq \vartheta \cup \operatorname{dom}\mathtt{L}$, so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C} \in \mathbb{X}_{\vartheta\cup\{y\}}$.

Moreover, y is bound by $\Box\mathscr{L}\{x\}[y\backslash u] = [z\backslash v]\mathtt{L}$, and $y \neq z$, since y is bound to u and z is bound to v. Hence y cannot occur free in the subterm $\mathtt{C}[\![z]\!]$ on the left-hand side of the step $\mathtt{r}$. In particular, y does not occur as a structural variable in $\mathtt{C}$. So applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we obtain that $\mathtt{C} \in \mathbb{X}_{\vartheta}$.

Then, adding an arbitrary substitution (Lem. 80) we have that $\mathtt{C}[z\backslash v\mathtt{L}] \in \mathbb{X}_{\vartheta}$, which contradicts the hypothesis that the step $\mathtt{r}$ is ϑ-internal.

C. **The context $\mathtt{C}_{11}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{11}[\mathtt{C}_1]$, so $\mathtt{C}_1[v] = w$, which is impossible.

(b) **The internal step $\mathtt{r}$ is to the left of $t_0 = t_0'[y\backslash u]$.** Let $\mathtt{r}_1 : t_0' \rightarrow_{\mathtt{sh\backslash gc}} \mathtt{C}_1[\![x]\!]$ be the step isomorphic to $\mathtt{r}$ but going under the context $[y\backslash u]$. Note that $\mathtt{r}_1$ must be $(\vartheta\cup\{y\})$-internal, otherwise $\mathtt{r}$ would be $(\vartheta\cup\{y\})$-external. By *i.h.* there is an evaluation context $\mathtt{C}_{10} \in \mathbb{X}_{\vartheta\cup\{y\}}$ such that $t_0' = \mathtt{C}_{10}[\![x]\!]$. It is immediate to conclude by taking $\mathtt{C}_0 := \mathtt{C}_{10}[y\backslash u] \in \mathbb{X}_{\vartheta}$, since then $t_0 = \mathtt{C}_{10}[\![x]\!][y\backslash u]$.

(c) **The internal step $\mathtt{r}$ is to the right of $t_0 = \mathtt{C}_1[\![x]\!][y\backslash t'_0]$.** Let $\mathtt{r}_1 : t'_0 \rightarrow_{\mathtt{sh\backslash gc}} u$. We consider two cases, depending on whether the step $\mathtt{r}_1$ is ϑ-internal or ϑ-internal:

i. **If $\mathtt{r}_1$ is ϑ-external.** Two further subcases, depending on whether y is a structural variable in $\mathtt{C}_1$ or not:

A. **If $y \in \mathsf{sv}(\mathtt{C}_1)$.** Since $\mathtt{r}_1 : t'_0 \rightarrow_{\mathtt{sh\backslash gc}} u$ is a ϑ-external step, we can write $t'_0 = \mathtt{C}_3[\Sigma]$ and $u = \mathtt{C}_3[\Sigma']$ where:

- Σ is the anchor of a redex, and Σ' its contractum,
- $\mathtt{C}_3$ is an evaluation context $\mathtt{C}_3 \in \mathcal{C}^h_\vartheta$.

Moreover, by the fact that structural variables are below evaluation contexts (Lem. 88) there exists an evaluation context $\mathtt{C}_2 \in \mathbb{X}_\vartheta$ such that $\mathtt{C}_1[\![x]\!] = \mathtt{C}_2[\![y]\!]$. Hence the step $\mathtt{r}$ is of the form:

$$\mathtt{r} : \mathtt{C}_2[\![y]\!][y\backslash \mathtt{C}_3[\Sigma]] \rightarrow_{\mathtt{sh\backslash gc}} \mathtt{C}_2[\![y]\!][y\backslash \mathtt{C}_3[\Sigma']]$$

If $\mathtt{C}_3$ happens to be a non-answer evaluation context, *i.e.* $\mathtt{C}_3 \in \mathcal{C}^{\cdot}_\vartheta$ then the composition $\mathtt{C}_2[\![y]\!][y\backslash \mathtt{C}_3]$ is a ϑ-evaluation context and $\mathtt{r}$ is a ϑ-external step, contradicting the hypothesis that it was internal.

So we may suppose that $\mathtt{C}_3$ is *not* a non-answer evaluation context. By Lem. 45 we know that evaluation contexts which are not non-answer evaluation contexts have the shape of an answer, that is, $\mathtt{C}_3[*]$ is an answer when filling the hole with an arbitrary term. In particular, $\mathtt{C}_3[\Sigma'] = u$ is both an answer and a structure, which is impossible.

B. **If $y \notin \mathsf{sv}(\mathtt{C}_1)$.** By the fact that non-structural variables are not required in "ϑ" (Lem. 87), we have that $\mathtt{C}_1 \in \mathbb{X}_\vartheta$. Then, regardless of whether t'_0 is a structure or not, adding an arbitrary substitution (Lem. 80) we have $\mathtt{C}_1[y\backslash t'_0] \in \mathbb{X}_\vartheta$. It is then immediate to conclude by taking $\mathtt{C}_0 := \mathtt{C}_1[y\backslash t'_0] \in \mathbb{X}_\vartheta$, since indeed $t_0 = \mathtt{C}_1[\![x]\!][y\backslash t'_0]$.

ii. **If $\mathtt{r}_1$ is ϑ-internal.** Then by the fact that structures are backwards preserved by internal steps (Lem. 104) we have that $t'_0 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$. We conclude by taking $\mathtt{C}_0 := \mathtt{C}_1[y\backslash t'_0] \in \mathbb{X}_\vartheta$, since $t_0 = \mathtt{C}_1[\![x]\!][y\backslash t'_0]$, as required.

5. **ESubsR, $\mathtt{C} = \mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}_2]$ where $\mathtt{C}_1 \in \mathbb{X}_\vartheta$ and $\mathtt{C}_2 \in \mathcal{C}^{\cdot}_\vartheta$.** We consider three cases, depending on whether (1) the internal step $\mathtt{r}$ is at the root of t_0, (2) t_0 is a substitution $t'_0[y\backslash r_0]$ and the step $\mathtt{r}$ is internal to t'_0, (3) t_0 is a substitution $t'_0[y\backslash r_0]$ and the step $\mathtt{r}$ is internal to r_0.

(a) **The internal step $\mathtt{r}$ is at the root of t_0.** Note that $\mathtt{r}$ cannot be a $\mathtt{dB}$ redex, since it would be external. So $\mathtt{r}$ is a $\mathtt{lsv}$ redex of the form:

$$\mathtt{r} : t_0 = \mathtt{C}[\![z]\!][z\backslash v\mathtt{L}] \rightarrow \mathtt{C}[v][z\backslash v]\mathtt{L} = \mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}_2[\![x]\!]] = t_1$$

Let $\mathtt{L}_1$ be the substitution context such that $\mathtt{L}_1[y\backslash \mathtt{C}_2[\![x]\!]] = [z\backslash v]\mathtt{L}$, and using Lem. 95 let us strip the substitution $\mathtt{L}_1$ from $\mathtt{C}_1[\![y]\!]$. This gives us two possibilities, **A** and **B**:

i. **Case A.** Then:

$$\mathtt{C}_1 = \mathtt{C}_{11}\mathtt{L}_1 \qquad \mathtt{C}[v] = \mathtt{C}_{11}[\![y]\!]$$

where $\hat{\vartheta} = \mathsf{fz}^\vartheta(\mathtt{L}_1)$ and $\mathtt{C}_{11} \in \mathbb{X}_{\hat{\vartheta}}$.

We consider three further subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{11}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{11}$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}_{11} \qquad \widehat{\mathtt{C}}[y, \Box] = \mathtt{C}$$

Note that y is bound by the substitution context $\mathtt{L}_1[y \backslash \mathtt{I}^{\vartheta}[\![x]\!]] = [z \backslash v]\mathtt{L}$ on the right-hand side of the step $\mathtt{r}$. So it must be the case that $y = z$, for if we had $y \in \operatorname{dom} \mathtt{L}$, we would have that y is free on the left-hand side of the step $\mathtt{r}$, since it occurs outside the scope of the substitution $\mathtt{L}$. This is impossible, since a free variable cannot become bound along reduction.

Also note that y is bound to $\mathtt{I}^{\vartheta}[\![x]\!]$ and, since $y = z$, we have $\mathtt{I}^{\vartheta}[\![x]\!] = v$. This is impossible, since answers do not have variables below non-answer evaluation contexts (Lem. 97).

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{11}$.** Then by the decomposition of evaluation contexts lemma (Lem. 50) the context $\mathtt{C}$ must be an evaluation context in $\mathbb{X}_{\hat{\vartheta}}$. Since $\hat{\vartheta} \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}$, by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C} \in \mathbb{X}_{\vartheta}$. This in turn implies that $\mathtt{C}[z \backslash v\mathtt{L}] \in \mathbb{X}_{\vartheta}$, contradicting the fact that the step $\mathtt{r}$ is ϑ-internal.

C. **The context $\mathtt{C}_{11}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{11}[\mathtt{C}_1]$, so $\mathtt{C}_1[v] = y$, which is impossible.

ii. **Case B.** Then:

$$\mathtt{C}_1 = \mathtt{C}_{11}[\![w]\!]\mathscr{L}\{\Box\} \qquad \mathtt{C}[v] = \mathtt{C}_{11}[\![w]\!] \qquad \mathtt{L}_1 = \Box\mathscr{L}\{y\}$$

where $\hat{\vartheta} = \mathtt{fz}^{\vartheta}(\mathtt{L}_1)$, the evaluation context $\mathtt{C}_{11}$ is in $\mathbb{X}_{\hat{\vartheta}}$, and $\mathscr{L}$ is a (ϑ, w)-chain context.

We consider three further subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{11}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{11}$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}_{11} \qquad \widehat{\mathtt{C}}[w, \Box] = \mathtt{C}$$

Note that w must be bound by the substitution context $\Box\mathscr{L}\{y\}[y \backslash \mathtt{I}^{\vartheta}[\![x]\!]] = [z \backslash v]\mathtt{L}$, since $\mathscr{L}$ is a (ϑ, w)-chain context. So it must be the case that $w = z$, for if we had that $w \in \operatorname{dom} \mathtt{L}$, it would be free on the left-hand side of the step $\mathtt{r}$, since w occurs outside the scope of $\mathtt{L}$. This is impossible, since free variables cannot become bound by reduction.

Note that w is bound by the substitution context $\Box\mathscr{L}\{y\}$ to a term of the form $\mathtt{I}_1^{\vartheta_1}[\![w_1]\!]$. Moreover, given that $w = z$, we have that $\mathtt{I}_1^{\vartheta_1}[\![w_1]\!] = v$. This is impossible, since answers do not have variables below non-answer evaluation contexts (Lem. 97).

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{11}$.** Then by the decomposition of evaluation contexts lemma (Lem. 50) the context $\mathtt{C}$ must be an evaluation context in $\mathbb{X}_{\hat{\vartheta}}$. Since $\hat{\vartheta} \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}$, by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C} \in \mathbb{X}_{\vartheta}$. Hence by adding an arbitrary substitution (Lem. 80), we obtain that $\mathtt{C}[z \backslash v\mathtt{L}] \in \mathbb{X}_{\vartheta}$. This contradicts the fact that $\mathtt{r}$ is ϑ-internal.

C. **The context $\mathtt{C}_{11}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{11}[\mathtt{C}_1]$, so $\mathtt{C}_1[v] = w$, which is impossible.

(b) **The internal step $\mathtt{r}$ is to the left of $t_0 = t'_0[y\backslash\mathtt{I}^\vartheta]$.** Let $\mathtt{r}_1 : t'_0 \to_{\mathtt{sh}\backslash\mathtt{gc}} \mathtt{C}_1[\![y]\!]$ be the step isomorphic to $\mathtt{r}$ but going under the context $[y\backslash\mathtt{I}^\vartheta[\![x]\!]]$. Note that $\mathtt{r}_1$ must be ϑ-internal, for if it were ϑ-external, by adding an arbitrary substitution (Lem. 80) it would contradict the fact that $\mathtt{r}$ is ϑ-internal.

So we may apply the *i.h.* to obtain that there exists an evaluation context $\mathtt{C}_{10} \in \mathbb{X}_\vartheta$ such that $t'_0 = \mathtt{C}_{10}[\![y]\!]$. Applying the ESubsR rule and taking $\mathtt{C}_0 := \mathtt{C}_{10}[\![y]\!][y\backslash\mathtt{I}^\vartheta] \in \mathbb{X}_\vartheta$, we have that $t_0 = \mathtt{C}_{10}[\![y]\!][y\backslash\mathtt{I}^\vartheta[\![x]\!]]$, as required.

(c) **The internal step $\mathtt{r}$ is to the right of $t_0 = \mathtt{C}_1[\![y]\!][y\backslash t'_0]$.** Let $\mathtt{r}_1 : t'_0 \to_{\mathtt{sh}\backslash\mathtt{gc}} \mathtt{I}^\vartheta[\![x]\!]$ be the step isomorphic to $\mathtt{r}$ but going under the context $\mathtt{C}_1[\![y]\!][y\backslash\Box]$. Note that $\mathtt{r}_1$ cannot be ϑ-external, since then $\mathtt{r}$ would be ϑ-external. Hence $\mathtt{r}_1$ is ϑ-external, and we may apply the *i.h.* to obtain that there is a non-answer evaluation context $\mathtt{I}^\vartheta_0 \in \mathcal{C}^{\cdot}_\vartheta$ such that $t'_0 = \mathtt{I}^\vartheta_0[\![x]\!]$. Taking $\mathtt{C}_0 := \mathtt{C}_1[\![y]\!][y\backslash\mathtt{I}^\vartheta_0] \in \mathbb{X}_\vartheta$, we have that $t_0 = \mathtt{C}_1[\![y]\!][y\backslash\mathtt{I}^\vartheta_0[\![x]\!]]$, as required.

6. **EAppRStr, $\mathtt{C} = u\,\mathtt{C}_1$ where $u \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ and $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$.** We consider three cases, depending on whether (1) the internal step $\mathtt{r}$ is at the root of t_0, (2) t_0 is an application $t'_0\,r_0$ and the step $\mathtt{r}$ is internal to t'_0, (3) t_0 is an application $t'_0\,r_0$ and the step $\mathtt{r}$ is internal to r_0.

(a) **The internal step $\mathtt{r}$ is at the root of t_0.** This case is impossible: $\mathtt{r}$ cannot be a $\mathtt{dB}$ step or a $\mathtt{lsv}$ step, since the right-hand side of both $\mathtt{dB}$ and $\mathtt{lsv}$ steps is a substitution, not an application.

(b) **The internal step $\mathtt{r}$ is to the left of $t_0 = t'_0\,\mathtt{C}_1[\![x]\!]$.** Let $\mathtt{r}_1 : t'_0 \to_{\mathtt{sh}\backslash\mathtt{gc}} u$ be the step isomorphic to $\mathtt{r}$ but going under the context $\Box\,\mathtt{C}_1[\![x]\!]$. Note that $\mathtt{r}_1$ must be ϑ-internal, otherwise $\mathtt{r}$ would be ϑ-external. Then by the fact that normal forms are backwards preserved by internal steps (Lem. 104), we have that t'_0 must be a strong ϑ-structure u_0. By taking $\mathtt{C}_0 := u_0\,\mathtt{C}_1 \in \mathbb{X}_\vartheta$ we have that $t_0 = u_0\,\mathtt{C}_1[\![x]\!]$, as required.

(c) **The internal step $\mathtt{r}$ is to the right of $t_0 = u\,t'_0$.** Let $\mathtt{r}_1 : t'_0 \to_{\mathtt{sh}\backslash\mathtt{gc}} \mathtt{C}_1[\![x]\!]$ be the step isomorphic to $\mathtt{r}$ but going under the context $u\,\Box$. By *i.h.* there is an evaluation context $\mathtt{C}_{10} \in \mathcal{C}^h_\vartheta$ such that $t'_0 = \mathtt{C}_{10}[\![x]\!]$. Taking $\mathtt{C}_0 := u\,\mathtt{C}_{10} \in \mathbb{X}_\vartheta$ we have that $t_0 = u\,\mathtt{C}_{10}[\![x]\!]$, as required.

7. **ELam, $\mathtt{C} = \lambda y.\mathtt{C}_1$, where $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{y\}}$.** Then t_0 is an abstraction $\lambda y.t'_0$ and the step $\mathtt{r}$ is internal to t'_0. Let $\mathtt{r}_1 : t'_0 \to_{\mathtt{sh}\backslash\mathtt{gc}} \mathtt{C}_1[\![x]\!]$ be the step isomorphic to $\mathtt{r}$ but going under the lambda. Note that $\mathtt{r}_1$ cannot be a $(\vartheta \cup \{y\})$-external step, since then $\mathtt{r}$ would be ϑ-external. Hence $\mathtt{r}_1$ is $(\vartheta \cup \{y\})$-internal and by *i.h.* we have that there exists an evaluation context $\mathtt{C}_{10} \in \mathcal{C}^h_{\vartheta\cup\{y\}}$ such that $t'_0 = \mathtt{C}_{10}[\![x]\!]$. Taking $\mathtt{C}_0 := \lambda y.\mathtt{C}_{10} \in \mathbb{X}_\vartheta$ we have that $t_0 = \lambda y.\mathtt{C}_{10}[\![x]\!]$, as required.

8. **eAppRCons, $\mathtt{C} = t\,\mathtt{C}_1$ and $t \in \mathcal{K}_\vartheta$ and $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$.** Similar to EAppRStr.

9. **eCase1. $\mathtt{C} = \mathtt{case}\ \mathtt{C}_1\ \mathtt{of}\ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I} \in \mathcal{C}^{\cdot}_\vartheta$ and $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$ and $h \notin \{\mathbf{c}_i\}_{i\in I}$ or $h = \mathbf{c}_j \in \{\mathbf{c}_i\}_{i\in I}$ and $|\mathtt{A}(\mathtt{C}, y)| \neq |\bar{x_j}|$.** Similar to previous cases.

10. **eCase2, $\mathtt{C} = \mathtt{case}\ t\ \mathtt{of}\ \mathbf{c}_1\bar{x_1} \Rightarrow t_1, \ldots, \mathbf{c}_j\bar{x_j} \Rightarrow \mathtt{C}_1, \ldots, \mathbf{c}_n\bar{x_n} \Rightarrow t_n \in \mathcal{C}^{\cdot}_\vartheta$ and $t \in \mathcal{N}_\vartheta$ and $t \not\succ (\mathbf{c}_i\bar{x}_i \Rightarrow s_i)_{i\in I}$ and $t_k \in \mathcal{N}_{\vartheta\cup\bar{x_k}}$ for all $k < j$ $\quad \mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\bar{x_i}}$.** Similar to previous cases.

$\square$

Lemma 106 (Permutation of internal steps and external $\mathtt{dB}$ steps)**.** *Given any set of variables ϑ such that $\mathsf{fv}(t_0) \subseteq \vartheta$, if $t_0 \rightarrow_{\mathtt{sh}}^{\neg\vartheta} t_1 \rightarrowtail_{\mathtt{sh}}^{\vartheta} t_3$ where the second step is a $\mathtt{dB}$ step, there exists a term t_2 such that $t_0 \rightarrowtail_{\mathtt{sh}}^{\vartheta} t_2 \twoheadrightarrow_{\mathtt{sh}\backslash\mathtt{gc}} t_3$ where the first step is a $\mathtt{dB}$ step. An explicit construction for the diagrams is given.*

Proof. Let $\mathtt{r}$ be the internal step $t_0 \rightarrow_{\mathtt{sh}}^{\neg\vartheta} t_1$ and $\mathtt{r}'$ the external $\mathtt{dB}$ step $t_1 \rightarrowtail_{\mathtt{sh}}^{\vartheta} t_3$. Then $t_1 = \mathtt{C}[(\lambda x.s)\mathtt{L}\,u]$ and $t_3 = \mathtt{C}[s[x\backslash u]\mathtt{L}]$. Throughout the proof we write Δ for the $\mathtt{dB}$ redex $(\lambda x.s)\mathtt{L}\,u$ and Δ' for its contractum $s[x\backslash u]\mathtt{L}$. By induction on the derivation that $\mathtt{C} \in \mathcal{C}_{\vartheta}^{h}$, the term t_0 will be shown to be of the form $\mathtt{C}_0[\Delta_0]$, where $\mathtt{C}_0 \in \mathcal{C}_{\vartheta}^{h}$, and Δ_0 is a $\mathtt{dB}$ redex, and $t_2 = \mathtt{C}_0[\Delta_0']$, where Δ_0' is the contractum of Δ_0, in such a way that the diagram is closed as required by the statement.

1. **EBox, $\mathtt{C} = \square \in \mathcal{C}_{\vartheta}^{h}$.** Then there is a $\mathtt{dB}$ redex at the root of t_1. By Lem. 79, the internal step $t_0 \rightarrow_{\mathtt{sh}}^{\neg\vartheta} t_1$ must be of the form

$$\mathtt{r} : t_0 = (\lambda x.s_0)\mathtt{L}_0\,u_0 \rightarrow_{\mathtt{sh}}^{\neg\vartheta} (\lambda x.s)\mathtt{L}\,u = t_1$$

   and the anchor of $\mathtt{r}$ must lie either inside s_0, inside u_0, or inside one of the arguments of $\mathtt{L}_0$. Then the situation is:

$$\begin{tikzcd}
(\lambda x.s_0)\mathtt{L}_0\,u_0 \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & (\lambda x.s)\mathtt{L}\,u \arrow[d, tail, "\vartheta"] \\
s_0[x\backslash u_0]\mathtt{L}_0 \arrow[r, "\mathtt{sh}\backslash\mathtt{gc}"'] & s[x\backslash u]\mathtt{L}
\end{tikzcd}$$

   By taking $\mathtt{C}_0 := \square$ we conclude.
   As a further observation, note that the step at the bottom of the diagram is not necessarily ϑ-internal, for instance, it is external if it happens to take place at the root of s_0.

2. **EAppL, $\mathtt{C} = \mathtt{C}_1\,r \in \mathcal{C}_{\vartheta}^{h}$ with $\mathtt{C}_1 \in \mathcal{C}_{\vartheta}^{h}$ and $h \neq \lambda$.** The situation is:

$$\begin{tikzcd}
t_0 \arrow[r, "\neg\vartheta"] & \mathtt{C}_1[\Delta]\,r \arrow[d, tail, "\vartheta"] \\
 & \mathtt{C}_1[\Delta']\,r
\end{tikzcd}$$

   We consider three cases: (1) the step $\mathtt{r}$ is at the root of t_0; (2) t_0 is an application $t_0 = t_0'\,r_0$ and the step $\mathtt{r}$ takes place inside t_0'; (3) t_0 is an application $t_0 = t_0'\,r_0$ and the step $\mathtt{r}$ takes place inside r_0.

   (a) **The internal step $\mathtt{r}$ is at the root of t_0.** We claim that this case is impossible. First, $\mathtt{r}$ cannot be a $\mathtt{dB}$ or a $\mathtt{case}$ step, since that would be a ϑ-external step, as $\square \in \mathcal{C}_{\vartheta}^{h}$. Second, $\mathtt{r}$ cannot be a $\mathtt{lsv}$ or a $\mathtt{fix}$ step, since its right-hand side is $\mathtt{C}_1[\Delta]\,r$, which is an application node, and the right-hand side of any $\mathtt{lsv}$ or $\mathtt{fix}$ step is a substitution node.

   (b) **The internal step $\mathtt{r}$ is to the left of $t_0 = t_0'\,r_0$.** Then there is a step $\mathtt{r}_1 : t_0' \rightarrow_{\mathtt{sh}\backslash\mathtt{gc}} \mathtt{C}_1[\Delta]$. We consider two subcases, depending on whether $\mathtt{r}_1$ is ϑ-external or ϑ-internal.

i. **If $\mathtt{r}_1$ is ϑ-external.** Then t'_0 is of the form $\mathtt{C}_2[\Sigma]$ where $\mathtt{C}_2 \in \mathcal{C}_\vartheta^{h'}$ and Σ is the anchor of a redex. Note that $h' = \lambda$, since otherwise $\mathtt{C}_2\, r \in \mathcal{C}_\vartheta^{h}$ and we would have that the step $\mathtt{r} : \mathtt{C}_2[\Sigma]\, r \to_{\mathtt{sh}}^{\neg\vartheta} \mathtt{C}_1[\Delta]\, r$ is external, contradicting the hypothesis that it is internal. Hence since $\mathtt{C}_2 \in \mathcal{C}_\vartheta^{\lambda}$ by Lem. 45 we conclude that t'_0 is of the form $v_0\mathtt{L}_0$, *i.e.* an answer. Moreover, since answers are stable by reduction (Lem. 77) we have that $\mathtt{C}_1[\Delta]$ is an answer, and this is impossible since answers do not have redexes below non-answer evaluation contexts (Lem. 97).

ii. **If $\mathtt{r}_1$ is ϑ-internal.** Then by *i.h.* we have that there exists a non-answer evaluation context $\mathtt{C}_2 \in \mathcal{C}_\vartheta^{h}$, a $\mathtt{dB}$ redex Δ_0, and Δ'_0 its contractum such that $t'_0 = \mathtt{C}_2[\Delta_0]$ and:

$$\begin{tikzcd} \mathtt{C}_2[\Delta_0] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_1[\Delta] \arrow[d, tail, "\vartheta"] \\ \mathtt{C}_2[\Delta'_0] \arrow[r, two heads, "\mathtt{sh}\backslash\mathtt{gc}"'] & \mathtt{C}_1[\Delta'] \end{tikzcd}$$

By taking $\mathtt{C}_0 := \mathtt{C}_2\, r \in \mathcal{C}_\vartheta^{h}$ we have that:

$$\begin{tikzcd} t_0 = \mathtt{C}_0[\Delta_0]\, r \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_1[\Delta]\, r = t_1 \arrow[d, tail, "\vartheta"] \\ t_2 = \mathtt{C}_0[\Delta'_0]\, r \arrow[r, two heads, "\mathtt{sh}\backslash\mathtt{gc}"'] & \mathtt{C}_1[\Delta']\, r = t_3 \end{tikzcd}$$

(c) **The internal step $\mathtt{r}$ is to the right of $t_0 = t'_0\, r_0$.** Then $t'_0 = \mathtt{C}_1[\Delta]$ and $r_0 \to_{\mathtt{sh}}^{\neg\vartheta} r$. By taking $\mathtt{C}_0 := \mathtt{C}_1\, r_0$, we have that $\mathtt{C}_0 \in \mathcal{C}_\vartheta^{h}$, and closing the diagram is immediate:

$$\begin{tikzcd} t_0 = \mathtt{C}_1[\Delta]\, r_0 \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_1[\Delta]\, r = t_1 \arrow[d, tail, "\vartheta"] \\ t_2 = \mathtt{C}_1[\Delta']\, r_0 \arrow[r, "\neg\vartheta"'] & \mathtt{C}_1[\Delta']\, r = t_3 \end{tikzcd}$$

3. **ESubLNonStr, $\mathtt{C} = \mathtt{C}_1[y\backslash r]$ with $y \notin \vartheta$, $r \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ and $\mathtt{C}_1 \in \mathcal{C}_\vartheta^{h}$.** The situation is:

$$\begin{tikzcd} t_0 \arrow[r, "\neg\vartheta"] & \mathtt{C}_1[\Delta][y\backslash r] \arrow[d, tail, "\vartheta"] \\ & \mathtt{C}_1[\Delta'][y\backslash r] \end{tikzcd}$$

There are three cases: (1) the step $\mathtt{r}$ is at the root of t_0; (2) t_0 is a substitution $t_0 = t'_0[y\backslash r_0]$ and the step $\mathtt{r}$ takes place inside t'_0; (3) t_0 is a substitution $t_0 = t'_0[y\backslash r_0]$ and the step $\mathtt{r}$ takes place inside r_0.

(a) **The internal step $\mathtt{r}$ is at the root of t_0.** Note that $\mathtt{r}$ cannot be a $\mathtt{dB}$, $\mathtt{fix}$ or $\mathtt{case}$

step, as that would be an external step. Suppose then that $\mathtt{r}$ is a $\mathtt{lsv}$ step:

$$\begin{array}{ccccc} t_0 = \mathtt{C}[\![z]\!][z\backslash v\mathtt{L}'] & \xrightarrow[\mathtt{r}]{\neg\vartheta} & \mathtt{C}[v][z\backslash v]\mathtt{L}' & = & \mathtt{C}_1[\Delta][y\backslash r] \\ & & & & \downarrow \vartheta \\ & & & & \mathtt{C}_1[\Delta'][y\backslash r] \end{array}$$

We know that $\mathtt{C}[v][z\backslash v]\mathtt{L}' = \mathtt{C}_1[\Delta][y\backslash r]$. The outermost substitution $[y\backslash r]$ is either $[z\backslash v]$ (if $\mathtt{L}'$ is empty) or it is the outermost substitution in $\mathtt{L}'$. In any case, the substitution $[y\backslash r]$ is not part of $\mathtt{C}$.

Let $\mathtt{L}_1$ be a substitution context such that $[z\backslash v]\mathtt{L}' = \mathtt{L}_1[y\backslash r]$ and using Lem. 95 let us strip the substitution $\mathtt{L}_1$ from $\mathtt{C}_1[\Delta]$. This gives us two possibilities, case **A** and case **B** in the statement of Lem. 95:

i. **Case A.** Then $\mathtt{C}_1 = \mathtt{C}_{11}\mathtt{L}_1$ in such a way that:

$$\mathtt{C}[v] = \mathtt{C}_{11}[\Delta]$$

where $\vartheta' = \mathsf{fz}^{\vartheta}(\mathtt{L}_1[y\backslash r]) = \mathsf{fz}^{\vartheta}([z\backslash v]\mathtt{L}')$ and $\mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta'}$.

We consider three subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{11}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{11}$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that

$$\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}_{11} \quad \text{and} \quad \widehat{\mathtt{C}}[\Delta, \Box] = \mathtt{C}$$

And the situation is:

$$\begin{array}{ccc} t_0 = \widehat{\mathtt{C}}[\Delta, z][z\backslash v\mathtt{L}'] & \xrightarrow{\neg\vartheta} & \widehat{\mathtt{C}}[\Delta, v][z\backslash v]\mathtt{L}' = t_1 \\ & & \downarrow \vartheta \\ & & \widehat{\mathtt{C}}[\Delta', v][z\backslash v]\mathtt{L}' = t_3 \end{array}$$

Recall that $\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta'}$ where $\vartheta' = \mathsf{fz}^{\vartheta}([z\backslash v]\mathtt{L}')$. Note that $z \notin \vartheta'$ since the value v is not a strong structure. By the fact that evaluation contexts are backwards-stable by substitutions (Lem. 89) there are two possibilities: the left and the right branch of the disjunction. Let us analyze each branch:

- **Left branch: $\widehat{\mathtt{C}}[\Box, z] \in \mathcal{C}^h_{\vartheta'}$.** Then by Lem. 91 we have that $\widehat{\mathtt{C}}[\Box, z][z\backslash v\mathtt{L}'] \in \mathbb{X}_{\vartheta}$, and the situation is:

$$\begin{array}{ccc} t_0 = \widehat{\mathtt{C}}[\Delta, z][z\backslash v\mathtt{L}'] & \xrightarrow{\neg\vartheta} & \widehat{\mathtt{C}}[\Delta, v][z\backslash v]\mathtt{L}' = t_1 \\ \downarrow \vartheta & & \downarrow \vartheta \\ t_2 = \widehat{\mathtt{C}}[\Delta', z][z\backslash v\mathtt{L}'] & \xrightarrow{\mathtt{sh}\backslash\mathtt{gc}} & \widehat{\mathtt{C}}[\Delta', v][z\backslash v]\mathtt{L}' = t_3 \end{array}$$

- **Right branch:** $\widehat{\mathtt{C}}[\Delta, \Box] \in \mathcal{C}^h_{\vartheta'}$**.** This case is not possible, as we would have that $\widehat{\mathtt{C}}[\Delta, \Box][z\backslash v]\mathtt{L}' \in \mathbb{X}_\vartheta$, since $\mathtt{fz}^\vartheta([z\backslash v]\mathtt{L}) = \vartheta'$ by Lem. 90. This implies that there are two different steps of the generalized call-by-need evaluation strategy under ϑ outgoing from t_1: one is the $\mathtt{dB}$ step:

$$t_1 = \widehat{\mathtt{C}}[\Delta, x][x\backslash v]\mathtt{L}' \rightarrowtail^\vartheta_{\mathtt{sh}} \widehat{\mathtt{C}}[\Delta', x][x\backslash v]\mathtt{L}' = t_3$$

and the other one is the $\mathtt{lsv}$ step:

$$t_1 = \widehat{\mathtt{C}}[\Delta, x][x\backslash v]\mathtt{L}' \rightarrowtail^\vartheta_{\mathtt{sh}} \widehat{\mathtt{C}}[\Delta, v][x\backslash v]\mathtt{L}'$$

The coexistence of two different steps contradicts the fact that $\rightarrowtail^\vartheta_{\mathtt{sh}}$ is a strategy (as shown in the unique decomposition lemma, Lem. 58).

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{11}$.** We claim that this case is impossible. Indeed, since $\mathtt{C}$ is a prefix of $\mathtt{C}_{11}$ we have that $\mathtt{C}_{11} = \mathtt{C}[\mathtt{C}']$ for some context $\mathtt{C}'$. By the decomposition of evaluation contexts (Lem. 50) the context $\mathtt{C}$ must be an evaluation context, more precisely $\mathtt{C} \in \mathcal{C}^h_{\vartheta'}$.
We claim that $\mathtt{C} \in \mathbb{X}_\vartheta$. Observe that $\vartheta' \subseteq \vartheta \cup \{z\} \cup \operatorname{dom} \mathtt{L}'$ since $\mathtt{C}_1 = \mathtt{C}_{11}\mathtt{L}_1$ and $\mathtt{L}_1[y\backslash r] = [z\backslash v]\mathtt{L}'$. But z is not bound to a structure in $[z\backslash v]$, so $z \notin \vartheta'$, which implies that $\vartheta' \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}'$. Moreover, by the variable convention, the context $\mathtt{C}$ cannot contain any occurrence of a variable in $\operatorname{dom} \mathtt{L}'$, since $\mathtt{C}$ takes part in an expression of the form $\mathtt{C}[\![z]\!][z\backslash v\mathtt{L}']$, in which it is outside the scope of $\mathtt{L}'$. In particular, the variables in $\operatorname{dom} \mathtt{L}'$ cannot be structural variables in $\mathtt{C}$, so by the fact that non-structural variables are not required in "ϑ" (Lem. 87), we conclude that indeed $\mathtt{C} \in \mathbb{X}_\vartheta$.
Recall that the internal step $\mathtt{r}$ is of the form:

$$t_0 = \mathtt{C}[\![z]\!][z\backslash v\mathtt{L}'] \rightarrow^{\neg\vartheta}_{\mathtt{sh}} \mathtt{C}[v][z\backslash v]\mathtt{L}' = t_1$$

but since $\mathtt{C}$ is a ϑ-evaluation context we conclude that the step $\mathtt{r}$ is actually external, which is a contradiction.

C. **The context $\mathtt{C}_{11}$ is a prefix of $\mathtt{C}$.** Since $\mathtt{C}_{11}$ is a prefix of $\mathtt{C}$ we have that $\mathtt{C} = \mathtt{C}_{11}[\mathtt{C}']$ for some context $\mathtt{C}'$. Hence $\mathtt{C}'[v] = (\lambda x.s)\mathtt{L}u$, and there are four possibilities for the position of the hole of $\mathtt{C}'$: inside s, inside one of the substitutions in $\mathtt{L}$, inside u, or right above $\lambda x.s$ (*i.e.* $\mathtt{C}' = \Box\mathtt{L}u$). Let us analyze each case:

- **The hole of $\mathtt{C}'$ is inside s.** Then $\mathtt{C}' = (\lambda x.\mathtt{C}'')\mathtt{L}u$ and $s = \mathtt{C}''[v]$. Then the steps are essentially orthogonal, *i.e.* $\Delta_\Phi = (\lambda x.\mathtt{C}''[\Phi])\mathtt{L}\,u$ and $\Delta'_\Phi = \mathtt{C}''[\Phi][x\backslash u]\mathtt{L}$; and the situation is:

$$\begin{tikzcd}
t_0 = \mathtt{C}_{11}[\Delta_z][z\backslash v\mathtt{L}'] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_{11}[\Delta_v][z\backslash v]\mathtt{L}' = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = \mathtt{C}_{11}[\Delta'_z][z\backslash v\mathtt{L}'] \arrow[r, "\mathtt{sh}\backslash\mathtt{gc}"] & \mathtt{C}_{11}[\Delta'_v][z\backslash v]\mathtt{L}' = t_3
\end{tikzcd}$$

To be able to close the diagram, we must justify that the context $\mathtt{C}_{11}[z\backslash v\mathtt{L}']$ is a ϑ-evaluation context. We already know that $\mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta'}$, and it suffices to show that $\mathtt{C}_{11} \in \mathbb{X}_\vartheta$.

Start by noting that $\vartheta' \subseteq \vartheta \cup \{z\} \cup \mathtt{L}'$ and $z \notin \vartheta'$ since z is bound to a value in $[z\backslash v]$, so actually $\vartheta' \subseteq \vartheta \cup \mathtt{L}'$. Moreover, the variables in $\operatorname{dom}\mathtt{L}'$ do not occur in the context $\mathtt{C}_{11}$ by the variable convention, since $\mathtt{C}_{11}$ takes part in an expression of the form $\mathtt{C}_{11}[(\lambda x.\mathtt{C}''[\![z]\!])\mathtt{L}u][z\backslash v\mathtt{L}']$ in which it is outside the scope of $\mathtt{L}'$. In particular, the variables in $\operatorname{dom}\mathtt{L}'$ cannot be structural variables in $\mathtt{C}_{11}$, so by the fact that non-structural variables are not required in "ϑ" (Lem. 87), we conclude that indeed $\mathtt{C}_{11} \in \mathbb{X}_{\vartheta}$.

- **The hole of $\mathtt{C}'$ is inside $\mathtt{L}$.** Then $\mathtt{C}' = (\lambda x.s)\mathtt{L}_{21}[x'\backslash\mathtt{C}'']\mathtt{L}_{22}u$ and $\mathtt{L} = \mathtt{L}_{21}[x'\backslash\mathtt{C}''[v]]\mathtt{L}_{22}$. Then the steps are essentially orthogonal, *i.e.* $\Delta_\Phi = (\lambda x.s)\mathtt{L}_{21}[x'\backslash\mathtt{C}''[\Phi]]\mathtt{L}_{22}\,u$ and $\Delta'_\Phi = s[x\backslash u]\mathtt{L}_{21}[x'\backslash\mathtt{C}''[\Phi]]\mathtt{L}_{22}$ and the diagram is closed as in the previous case.
- **The hole of $\mathtt{C}'$ is inside u.** Then $\mathtt{C}' = (\lambda x.s)\mathtt{L}\mathtt{C}''$ and $u = \mathtt{C}''[v]$. Then the steps are essentially orthogonal, *i.e.* $\Delta_\Phi = (\lambda x.s)\mathtt{L}\,\mathtt{C}''[\Phi]$ and $\Delta'_\Phi = s[x\backslash\mathtt{C}''[\Phi]]\mathtt{L}$ and the diagram is closed as in the previous case.
- **The context $\mathtt{C}'$ is of the form $\mathtt{C}' = \Box\mathtt{L}u$.** We claim that this case is impossible.
  Observe that in this case the internal step $\mathtt{r}$ is of the form:
  $$t_0 = \mathtt{C}_{11}[z\mathtt{L}s][z\backslash v\mathtt{L}'] \to^{\neg\vartheta}_{\mathtt{sh}} \mathtt{C}_{11}[v\mathtt{L}s][z\backslash v]\mathtt{L}' = t_1$$
  As in the previous case, we may argue that $\mathtt{C}_{11} \in \mathbb{X}_{\vartheta}$. This in turn implies that $\mathtt{C}_{11}[\Box\mathtt{L}s][z\backslash v\mathtt{L}'] \in \mathbb{X}_{\vartheta}$. This means that the $\mathtt{r}$ is actually an external step, which is a contradiction.

ii. **Case B.** Then $\mathtt{C}_1 = \mathtt{C}_{11}[\![x']\!]\mathscr{L}\{\Box\}$ in such a way that:
$$\mathtt{C}[v] = \mathtt{C}_{11}[\![x']\!] \qquad \mathtt{L}_1 = \Box\mathscr{L}\{\Delta\}$$
where $\vartheta' = \mathsf{fz}^{\vartheta}(\mathtt{L}_1[y\backslash r]) = \mathsf{fz}^{\vartheta}([z\backslash v]\mathtt{L}')$, the evaluation context is $\mathtt{C}_{11} \in \mathcal{C}^{h}_{\vartheta'}$, and $\mathscr{L}$ is a (ϑ, x')-chain context.
We consider three subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{11}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{11}$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:
$$\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}_{11} \qquad \widehat{\mathtt{C}}[x', \Box] = \mathtt{C}$$
By the fact that evaluation contexts are backwards-stable by substitutions (Lem. 89) there are two possibilities: the left and the right branch of the disjunction. Let us analyze each branch:
- **Left branch: $\widehat{\mathtt{C}}[\Box, z] \in \mathtt{C}_{11}$.** Note that $\mathscr{L}$ is a (ϑ, x')-chain context, so $\Box\mathscr{L}\{\Delta\}$ binds x'. Moreover, $\Box\mathscr{L}\{\Delta\}[y\backslash r] = \mathtt{L}_1[y\backslash r] = [z\backslash v]\mathtt{L}'$. So it must be the case that $x' = z$. Otherwise, x' would be bound by a substitution in $\mathtt{L}'$ and it would occur free on the left hand side $t_0 = \widehat{\mathtt{C}}[x', z][z\backslash v\mathtt{L}']$ which is impossible since a free variable cannot become bound by reduction.

  Then since $x' = z$ we have that $v\mathtt{L}' = \mathtt{C}_1[\Delta]$ for some evaluation context $\mathtt{I}^{\vartheta} \in \mathcal{C}^{\cdot}_{\vartheta}$. This is impossible by Lem. 97.

- **Right branch:** $\widehat{\mathtt{C}}[x', \Box] \in \mathtt{C}_{11}$**.** Then $\widehat{\mathtt{C}}[x', \Box] \in \mathcal{C}^h_{\vartheta'}$. Note that $\vartheta' = \mathtt{fz}^{\vartheta}([z\backslash v]\mathtt{L}')$, so $\vartheta' \subseteq \vartheta \cup \mathrm{dom}\,\mathtt{L}'$. Since variables in $\mathrm{dom}\,\mathtt{L}'$ do not occur in $\mathtt{C}$, as $\mathtt{C}$ is outside the scope of $\mathtt{L}'$ on the starting term t_0, in particular they are not structural variables, so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\widehat{\mathtt{C}}[x', \Box] \in \mathcal{C}^h_{\vartheta}$, so $\mathtt{C}[z\backslash v\mathtt{L}'] = \widehat{\mathtt{C}}[x', \Box][x\backslash v\mathtt{L}']$ is an evaluation context in $\mathcal{C}^h_{\vartheta}$. This means that the step $\mathtt{r}$ is internal, which is a contradiction.

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{11}$.** Then $\mathtt{C}_{11} = \mathtt{C}[\mathtt{C}']$, so by the decomposition lemma for evaluation contexts (Lem. 50) $\mathtt{C} \in \mathcal{C}^h_{\vartheta'}$. Note that $\vartheta' \subseteq \vartheta \cup \mathrm{dom}\,\mathtt{L}$, so by the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$. This means that the step $\mathtt{r}$ is internal, contradicting the hypothesis.

C. **The context $\mathtt{C}_{11}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{11}[\mathtt{C}']$, so since $\mathtt{C}[v] = \mathtt{C}_{11}[\![x']\!]$ we have that $v = x'$ which is impossible.

(b) **The internal step $\mathtt{r}$ is to the left of $t_0 = t'_0[y\backslash r_0]$.** Then $r_0 = r$ and the internal step $\mathtt{r}$ is of the form:

$$t_0 = t'_0[y\backslash r] \to^{\neg\vartheta}_{\mathtt{sh}} t'_1[y\backslash r] = \mathtt{C}_1[\Delta][y\backslash r] = t_1$$

Note that the corresponding step $t'_0 \to_{\mathtt{sh}\backslash\mathtt{gc}} t'_1$ is internal:

$$t'_0 \to^{\neg\vartheta}_{\mathtt{sh}} t'_1 = \mathtt{C}_1[\Delta]$$

for, were it external, we would conclude that the step $\mathtt{r}$ is also external, contradicting the hypothesis. Moreover:

$$t'_1 = \mathtt{C}_1[\Delta] \rightarrowtail^{\vartheta}_{\mathtt{sh}} \mathtt{C}_1[\Delta'] = t'_3$$

By *i.h.* we have that there exists an evaluation context $\mathtt{C}_{10} \in \mathcal{C}^h_{\vartheta}$, a $\mathtt{dB}$ redex Δ_0, and Δ'_0 its contractum such that $t'_0 = \mathtt{C}_{10}[\Delta_0]$ and:

$$\begin{tikzcd}
\mathtt{C}_{10}[\Delta_0] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_1[\Delta] \arrow[d, tail, "\vartheta"] \\
\mathtt{C}_{10}[\Delta'_0] \arrow[r, two heads, "\mathtt{sh}\backslash\mathtt{gc}"] & \mathtt{C}_1[\Delta']
\end{tikzcd}$$

By taking $\mathtt{C}_0 := \mathtt{C}_{10}[y\backslash r] \in \mathcal{C}^h_{\vartheta}$ we have that:

$$\begin{tikzcd}
t_0 = \mathtt{C}_0[\Delta_0][y\backslash r] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_1[\Delta][y\backslash r] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = \mathtt{C}_0[\Delta'_0][y\backslash r] \arrow[r, two heads, "\mathtt{sh}\backslash\mathtt{gc}"] & \mathtt{C}_1[\Delta'][y\backslash r] = t_3
\end{tikzcd}$$

(c) **The internal step $\mathtt{r}$ is to the right of $t_0 = t'_0[y\backslash r_0]$.** Then $t'_0 = \mathtt{C}_1[\Delta]$ and the internal step $\mathtt{r}$ is of the form:

$$t_0 = \mathtt{C}_1[\Delta][y\backslash r_0] \to^{\neg\vartheta}_{\mathtt{sh}} \mathtt{C}_1[\Delta][y\backslash r] = t_1$$

where $r_0 \to_{\mathtt{sh\backslash gc}} r$, and we may conclude directly, since the internal and the external steps are essentially orthogonal:

$$\begin{tikzcd}
t_0 = \mathtt{C}_1[\Delta][y\backslash r_0] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_1[\Delta][y\backslash r] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = \mathtt{C}_1[\Delta'][y\backslash r_0] \arrow[r, "\mathtt{sh\backslash gc}"] & \mathtt{C}_1[\Delta'][y\backslash r] = t_3
\end{tikzcd}$$

To be able to close the diagram, we must justify that the step at the left is external, *i.e.* that $\mathtt{C}_1[y\backslash r_0]$ is an ϑ-evaluation context. Indeed, the facts that $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$ and that adding an arbitrary substitution preserves evaluation contexts (Lem. 80) imply that $\mathtt{C}_1[y\backslash r_0] \in \mathcal{C}^h_\vartheta$, as required.

4. **ESubLStr, $\mathtt{C} = \mathtt{C}_1[y\backslash r]$ with $r \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ and $\mathtt{C}_1 \in \mathcal{C}^h_{\vartheta\cup\{y\}}$.** The situation is:

$$\begin{tikzcd}
t_0 \arrow[r, "\neg\vartheta"] & \mathtt{C}_1[\Delta][y\backslash r] = t_1 \arrow[d, tail, "\vartheta"] \\
 & \mathtt{C}_1[\Delta'][y\backslash r] = t_3
\end{tikzcd}$$

There are three cases: (1) the step $\mathtt{r}$ is at the root of t_0; (2) t_0 is a substitution $t'_0[y\backslash r_0]$ and the step $\mathtt{r}$ takes place inside t'_0; (3) t_0 is a substitution $t'_0[y\backslash r_0]$ and the step $\mathtt{r}$ takes place inside r_0.

(a) **The internal step $\mathtt{r}$ is at the root of t_0.** Note that $\mathtt{r}$ cannot be a $\mathtt{dB}$, $\mathtt{fix}$ or $\mathtt{case}$ step, as that would be an external step. Suppose then that $\mathtt{r}$ is a $\mathtt{lsv}$ step:

$$\begin{tikzcd}
t_0 = \mathtt{C}[\![z]\!][z\backslash v\mathtt{L}'] \arrow[r, "\neg\vartheta", "\mathtt{r}"'] & \mathtt{C}[v][z\backslash v]\mathtt{L}' \arrow[r, equal] & \mathtt{C}_1[\Delta][y\backslash r] = t_1 \arrow[d, tail, "\vartheta"] \\
 & & \mathtt{C}_1[\Delta'][y\backslash r] = t_3
\end{tikzcd}$$

We know that $\mathtt{C}[v][z\backslash v]\mathtt{L}' = \mathtt{C}_1[(\lambda x.s)\mathtt{L}\,u][y\backslash r]$. Note that $\mathtt{L}'$ cannot be empty since the outermost substitution $[y\backslash r]$ cannot coincide with $[z\backslash v]$, given that $r \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ is a structure, and therefore it cannot be a value like v.

Let $\mathtt{L}_1$ be a substitution context such that $[z\backslash v]\mathtt{L}' = \mathtt{L}_1[y\backslash r]$, and using Lem. 95 let us strip the substitution $\mathtt{L}_1$ from $\mathtt{C}_1[\Delta]$. This gives us two possibilities, case **A** and case **B** in the statement of Lem. 95:

i. **Case A.** Then $\mathtt{C}_1 = \mathtt{C}_{11}\mathtt{L}_1$ in such a way that

$$[z\backslash v]\mathtt{L}' = \mathtt{L}_1[y\backslash r] \qquad \text{and} \qquad \mathtt{C}[v] = \mathtt{C}_{11}[(\lambda x.s)\mathtt{L}\,u]$$

where $\vartheta' \cup \{y\} = \mathsf{fz}^{\vartheta\cup\{y\}}(\mathtt{L}_1[y\backslash r]) = \mathsf{fz}^{\vartheta\cup\{y\}}([z\backslash v]\mathtt{L}')$ and $\mathtt{C}_{11} \in \mathbb{X}_{\vartheta'\cup\{y\}}$.

We consider three subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{11}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{11}$ are disjoint.** Then there exists a two-hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}_{11} \quad \text{and} \quad \widehat{\mathtt{C}}[\Delta, \Box] = \mathtt{C}$$

Then the situation is:

$$\begin{tikzcd}
t_0 = \widehat{\mathtt{C}}[\Delta, z][z\backslash v\mathtt{L}'] \arrow[r, "\neg\vartheta"] & \widehat{\mathtt{C}}[\Delta, v][z\backslash v]\mathtt{L}' = t_1 \arrow[d, "\vartheta"] \\
 & \widehat{\mathtt{C}}[\Delta', v][z\backslash v]\mathtt{L}' = t_3
\end{tikzcd}$$

By the fact that evaluation contexts are backwards-stable by substitutions (Lem. 89) there are two possibilities for $\widehat{\mathtt{C}}$: the left and the right branch of the disjunction. Let us analyze each branch:

- **Left branch.** $\widehat{\mathtt{C}}[\Box, z] \in \mathbb{X}_{\vartheta' \cup \{y\}}$. Then by Lem. 91 we have that $\widehat{\mathtt{C}}[\Box, z][z\backslash v\mathtt{L}'] \in \mathbb{X}_{\vartheta}$, and the diagram can be closed:

$$\begin{tikzcd}
t_0 = \widehat{\mathtt{C}}[\Delta, z][z\backslash v\mathtt{L}'] \arrow[r, "\neg\vartheta"] \arrow[d, "\vartheta"'] & \widehat{\mathtt{C}}[\Delta, v][z\backslash v]\mathtt{L}' = t_1 \arrow[d, "\vartheta"] \\
t_2 = \widehat{\mathtt{C}}[\Delta', z][z\backslash v\mathtt{L}'] \arrow[r, "\mathtt{sh\backslash gc}"] & \widehat{\mathtt{C}}[\Delta', v][z\backslash v]\mathtt{L}' = t_3
\end{tikzcd}$$

- **Right branch.** $\widehat{\mathtt{C}}[\Delta, \Box] \in \mathbb{X}_{\vartheta' \cup \{y\}}$. This contradicts the fact that the step $\mathtt{r}$ at the top of the diagram is internal.

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{11}$.** Then $\mathtt{C}_{11} = \mathtt{C}[\mathtt{C}_1]$. By the decomposition of evaluation contexts lemma (Lem. 50) we obtain that $\mathtt{C} \in \mathbb{X}_{\vartheta' \cup \{y\}}$. This contradicts the fact that the step $\mathtt{r}$ at the top of the diagram is internal.

C. **The context $\mathtt{C}_{11}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{11}[\mathtt{C}_1]$, so the $\mathtt{dB}$ redex contracted by $\mathtt{r}'$ is $\Delta = (\lambda x.s)\mathtt{L}u = \mathtt{C}_1[v]$. There are four subcases, depending on the position of v inside Δ: it can be internal to s, internal to $\mathtt{L}$, internal to u or it can be precisely $\lambda x.s$:

- v **is internal to** s**.** That is, $\mathtt{C}_1 = (\lambda x.\mathtt{C}_2)\mathtt{L}u$ and $s = \mathtt{C}_2[v]$. Let $\Delta_\Phi = (\lambda x.\mathtt{C}_2[\Phi])\mathtt{L}u$ and $\Delta'_\Phi = \mathtt{C}_2[\Phi][x\backslash u]\mathtt{L}$. Then the diagram can be closed as follows:

$$\begin{tikzcd}
t_0 = \mathtt{C}_{11}[\Delta_z][z\backslash v\mathtt{L}'] \arrow[r, "\neg\vartheta"] \arrow[d, "\vartheta"'] & \mathtt{C}_{11}[\Delta_v][z\backslash v]\mathtt{L}' = t_1 \arrow[d, "\vartheta"] \\
t_2 = \mathtt{C}_{11}[\Delta'_z][z\backslash v\mathtt{L}'] \arrow[r, "\mathtt{sh\backslash gc}"] & \mathtt{C}_{11}[\Delta'_v][z\backslash v]\mathtt{L}' = t_3
\end{tikzcd}$$

- v **is internal to** $\mathtt{L}$**.** That is, $\mathtt{C}_1 = (\lambda x.s)\mathtt{L}_1[x'\backslash \mathtt{C}_2]\mathtt{L}_2 u$ and $\mathtt{L} = \mathtt{L}_1[x'\backslash \mathtt{C}_2[v]]\mathtt{L}_2$. Let $\Delta_\Phi = (\lambda x.s)\mathtt{L}_1[x'\backslash \mathtt{C}_2[\Phi]]\mathtt{L}_2 u$ and $\Delta'_\Phi = s[x\backslash u]\mathtt{L}_1[x'\backslash \mathtt{C}_2[\Phi]]\mathtt{L}_2$. Then the diagram can be closed as in the previous subcase.

- v **is internal to** u. That is, $\mathtt{C}_1 = (\lambda x.s)\mathtt{L}\mathtt{C}_2$ and $u = \mathtt{C}_2[v]$. Let $\Delta_\Phi = (\lambda x.s)\mathtt{L}\mathtt{C}_2[\Phi]$ and $\Delta'_\Phi = s[x\backslash\mathtt{C}_2[\Phi]]\mathtt{L}$. Then the diagram can be closed as in the previous subcase.
- v **is precisely** $\lambda x.s$. That is, $\mathtt{C}_1 = \Box\mathtt{L}u$. Then the step $\mathtt{r}$ at the top of the diagram is of the form:

$$t_0 = \mathtt{C}_{11}[z\mathtt{L}u][z\backslash v\mathtt{L}'] \to \mathtt{C}_{11}[v\mathtt{L}u][z\backslash v]\mathtt{L}' = t_1$$

  Note that $\mathtt{C}_{11}[z\backslash v\mathtt{L}'] \in \mathcal{C}^h_\vartheta$ by Lem. 91. Hence by the decomposition lemma for evaluation contexts lemma (Lem. 50) we have $\mathtt{C}_{11}[\Box\mathtt{L}u][z\backslash v\mathtt{L}'] \in \mathcal{C}^h_\vartheta$, which contradicts the fact that $\mathtt{r}$ is an internal step.

ii. **Case B.** Then $\mathtt{C}_1 = \mathtt{C}_{11}[\![x']\!]\mathscr{L}\{\Box\}$ such that:

$$\mathtt{C}[v] = \mathtt{C}_{11}[\![x']\!] \qquad \Box\mathscr{L}\{\Delta\} = \mathtt{L}_1$$

where $\vartheta' \cup \{y\} = \mathsf{fz}^{\vartheta\cup\{y\}}(\mathtt{L}_1)$, the evaluation context $\mathtt{C}_{11}$ is in $\mathbb{X}_{\vartheta'\cup\{y\}}$, and $\mathscr{L}$ is a $(\vartheta \cup \{y\}, x')$-chain context.
We consider three subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{11}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{11}$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}_{11} \qquad \widehat{\mathtt{C}}[x', \Box] = \mathtt{C}$$

We argue that this case is impossible. Note that the original term is of the form:

$$t_0 = \mathtt{C}[\![z]\!][z\backslash v\mathtt{L}'] = \widehat{\mathtt{C}}[x', v][z\backslash v\mathtt{L}'] = \mathtt{C}_{11}[\![x']\!][z\backslash v\mathtt{L}']$$

and $t_0 \to t_1$. The variable x' is bound in t_1, so it must also be bound in t_0, so $x' = z$. Since $\mathscr{L}$ is a $(\vartheta \cup \{y\}, x')$-chain context we know that $\Box_1\mathscr{L}\{\Box_2\} = \Box_1[z\backslash\Box_2]\mathtt{L}'$. This means that $v = \mathtt{I}_1^{\vartheta''}[\Delta]$ where $\mathtt{I}^{\vartheta''}$ is a non-answer evaluation context, for some appropiate value of ϑ''. In any case, this is impossible, since answers do not have redexes under non-answer evaluation contexts (Lem. 97).

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{11}$.** Then $\mathtt{C}_{11} = \mathtt{C}[\mathtt{C}']$. Hence by the decomposition lemma for evaluation contexts (Lem. 58) we have that $\mathtt{C}$ must be an evaluation context in $\mathbb{X}_{\vartheta'\cup\{y\}}$.
As in the two previous cases, we may note that $\vartheta' \cup \{y\} \subseteq y \cup \mathrm{dom}\,\mathtt{L}'$ and apply Lem. 87 to conclude that $\mathtt{C} \in \mathbb{X}_\vartheta$. So also $\mathtt{C}[z\backslash v\mathtt{L}] \in \mathbb{X}_\vartheta$, which contradicts that that the step $\mathtt{r}$ is internal.

C. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{11}$.** Then $\mathtt{C}_{11} = \mathtt{C}[\mathtt{C}']$. Since $\mathtt{C}[v] = \mathtt{C}_{11}[\![x']\!]$ we have that $v = x'$, which is impossible.

(b) **The internal step $\mathtt{r}$ is to the left of $t_0 = t'_0[y\backslash r]$.** Here $t'_0 \to_{\mathtt{sh}\backslash\mathtt{gc}} t'_1 = \mathtt{C}_1[\Delta]$. Note that the step $t'_0 \to_{\mathtt{sh}\backslash\mathtt{gc}} t'_1$ must be $(\vartheta \cup \{y\})$-internal, for otherwise the step at the top of the diagram $\mathtt{r} : t'_0[y\backslash r] \to_{\mathtt{sh}\backslash\mathtt{gc}} t'_1[y\backslash r]$ would be a ϑ-external step.

By *i.h.* we have that there exists an evaluation context $\mathtt{C}_{10} \in \mathbb{X}_{\vartheta\cup\{y\}}$, a $\mathtt{dB}$ redex Δ_0, and its contractum Δ'_0 such that $t'_0 = \mathtt{C}_{10}[\Delta_0]$ and:

$$\begin{tikzcd}
\mathtt{C}_{10}[\Delta_0] \arrow[r, "\neg(\vartheta\cup\{y\})"] \arrow[d, tail, "(\vartheta\cup\{y\})-\text{need}"'] & \mathtt{C}_1[\Delta] \arrow[d, tail, "(\vartheta\cup\{y\})-\text{need}"] \\
\mathtt{C}_{10}[\Delta'_0] \arrow[r, twoheadrightarrow, "\mathtt{sh}\backslash\mathtt{gc}"] & \mathtt{C}_1[\Delta']
\end{tikzcd}$$

By taking $\mathtt{C}_0 := \mathtt{C}_{10}[y\backslash r] \in \mathcal{C}^h_\vartheta$ we have:

$$\begin{tikzcd}
t_0 = \mathtt{C}_{10}[\Delta_0][y\backslash r] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_1[\Delta][y\backslash r] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = \mathtt{C}_{10}[\Delta'_0][y\backslash r] \arrow[r, twoheadrightarrow, "\mathtt{sh}\backslash\mathtt{gc}"] & \mathtt{C}_1[\Delta'][y\backslash r] = t_3
\end{tikzcd}$$

(c) **The internal step $\mathtt{r}$ is to the right of $t_0 = t'_0[y\backslash r_0]$.** Here $r_0 \to_{\mathtt{sh}\backslash\mathtt{gc}} r$ and $t'_0 = \mathtt{C}_1[\Delta]$. We consider two cases, depending on whether the step $\mathtt{r}_1 : r_0 \to_{\mathtt{sh}\backslash\mathtt{gc}} r$ is ϑ-external or ϑ-internal:

i. **If $\mathtt{r}_1$ is ϑ-external.** Two further subcases, depending on whether $y \in \mathsf{sv}(\mathtt{C}_1)$ or not:

- **If $y \in \mathsf{sv}(\mathtt{C}_1)$.** Since $\mathtt{r}_1 : r_0 \to_{\mathtt{sh}\backslash\mathtt{gc}} r$ is a ϑ-external step, we can write $r_0 = \mathtt{C}_3[\Sigma]$ and $r = \mathtt{C}_3[\Sigma']$ where:
  - Σ is the anchor of $\mathtt{r}_1$ and Σ' is its contractum,
  - $\mathtt{C}_3$ is an evaluation context $\mathtt{C}_3 \in \mathcal{C}^h_\vartheta$.

  Moreover, by the fact that structural variables are below evaluation contexts (Lem. 88) there is an evaluation context $\mathtt{C}_2 \in \mathcal{C}^h_\vartheta$ such that $\mathtt{C}_1[\Delta] = \mathtt{C}_2[\![y]\!]$. Hence the step $\mathtt{r}$ at the top of the diagram is of the form:

  $$\mathtt{r} : t_0 = \mathtt{C}_2[\![y]\!][y\backslash\mathtt{C}_3[\Sigma]] \to_{\mathtt{sh}\backslash\mathtt{gc}} \mathtt{C}_2[\![y]\!][y\backslash\mathtt{C}_3[\Sigma']] = t_1$$

  If $\mathtt{C}_3$ happens to be a non-answer evaluation context, *i.e.* $\mathtt{C}_3 \in \mathcal{C}^{\cdot}_\vartheta$ then the composition $\mathtt{C}_2[\![y]\!][y\backslash\mathtt{C}_3]$ is a ϑ-evaluation context and $\mathtt{r}$ is a ϑ-external step, contradicting the hypothesis that it was internal.
  So we may suppose that $\mathtt{C}_3$ is *not* a non-answer evaluation context. By Lem. 45 we know that evaluation contexts which are not non-answer evaluation contexts have the shape of an answer. In particular $\mathtt{C}_3[\Sigma'] = (\lambda x'.t')\mathtt{L}''$ and we have a ϑ-external step:

  $$\mathtt{r}_2 : t_1 = \mathtt{C}_2[\![y]\!][y\backslash(\lambda x'.t')\mathtt{L}''] \rightarrowtail^{\vartheta}_{\mathtt{sh}} \mathtt{C}_2[\lambda x'.t'][y\backslash\lambda x'.t']\mathtt{L}''$$

  Hence t_1 has two distinct external steps, namely $\mathtt{r}'$ and $\mathtt{r}_2$. This is impossible as a consequence of the unique decomposition lemma (Lem. 58).
- **If $y \notin \mathsf{sv}(\mathtt{C}_1)$.** Then by the fact that non-structural variables are not required in "ϑ" (Lem. 87), we have that $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$, so $\mathtt{C}_1[y\backslash r_0] \in \mathcal{C}^h_\vartheta$, regardless of whether

r_0 is a ϑ-structure or not. This lets us close the diagram as follows:

$$\begin{tikzcd}
t_0 = \mathtt{C}_1[\Delta][y\backslash r_0] \arrow[r, "\neg\vartheta"] \arrow[d, "\vartheta"'] & \mathtt{C}_1[\Delta][y\backslash r] = t_1 \arrow[d, "\vartheta"] \\
t_2 = \mathtt{C}_1[\Delta'][y\backslash r_0] \arrow[r, "\mathtt{sh}\backslash\mathtt{gc}"] & \mathtt{C}_1[\Delta'][y\backslash r] = t_3
\end{tikzcd}$$

ii. **If $\mathtt{r}_1$ is ϑ-internal.** Then by the fact that normal forms are backwards preserved by internal steps (Lem. 104), $\mathtt{r}_0$ is a structure; more precisely $\mathtt{r}_0 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$. This allows us to conclude that $\mathtt{C}_1[y\backslash r_0] \in \mathcal{C}^h_\vartheta$, and the diagram can be closed just like in the previous subcase.

5. **ESubsR, $\mathtt{C} = \mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}_2]$, where $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$ and $\mathtt{C}_2 \in \mathcal{C}^{\cdot}_\vartheta$.** The situation is:

$$\begin{tikzcd}
t_0 \arrow[r, "\neg\vartheta", "\mathtt{r}"'] & \mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}_2[\Delta]] = t_1 \arrow[d, "\vartheta"] \\
 & \mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}_2[\Delta']] = t_3
\end{tikzcd}$$

There are three cases: (1) the step $\mathtt{r}$ is at the root of t_0; (2) t_0 is a substitution $t'_0[y\backslash r_0]$ and the step takes place inside t'_0; (3) t_0 is a substitution $t'_0[y\backslash r_0]$ and the step takes place inside r_0.

(a) **The internal step $\mathtt{r}$ is at the root of t_0.** Note that $\mathtt{r}$ cannot be a $\mathtt{dB}$, $\mathtt{fix}$ or $\mathtt{case}$ step since then it would be ϑ-external, so $\mathtt{r}$ must be a $\mathtt{lsv}$ step:

$$t_0 = \mathtt{C}[\![z]\!][z\backslash v\mathtt{L}'] \xrightarrow{\neg\vartheta} \mathtt{C}[\![z]\!][z\backslash v]\mathtt{L}' = \mathtt{C}_1[\![y]\!][y\backslash \mathtt{C}_2[\Delta]] = t_1$$

Let $\mathtt{L}_1$ be a substitution context such that $[z\backslash v]\mathtt{L}' = \mathtt{L}_1[y\backslash \mathtt{C}_2[\Delta]]$, and using Lem. 95 let us strip the substitution $\mathtt{L}_1$ from $\mathtt{C}_1[\![y]\!]$. This gives us two possibilities, case **A** and case **B** in the statement of Lem. 95:

i. **Case A.** Then $\mathtt{C}_1 = \mathtt{C}_{11}\mathtt{L}_1$ such that:

$$\mathtt{C}_{11}[\![y]\!] = \mathtt{C}[v]$$

where $\vartheta' = \mathsf{fz}^\vartheta(\mathtt{L}_1)$ and the evaluation context $\mathtt{C}_{11} \in \mathcal{C}^{h'}_{\vartheta'}$.

We consider three subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{11}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{11}$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}_{11} \qquad \widehat{\mathtt{C}}[y, \Box] = \mathtt{C}$$

Then the original term t_0 is of the form:

$$t_0 = \mathtt{C}[\![z]\!][z\backslash v\mathtt{L}'] = \widehat{\mathtt{C}}[y, z][z\backslash v\mathtt{L}'] = \mathtt{C}_{11}[\![y]\!][z\backslash v\mathtt{L}']$$

and $t_0 \to_{\mathtt{sh\backslash gc}} t_1$. The variable y occurs bound in t_1, so it must also occur bound in t_0, which means that $y = z$. Since $\mathtt{L}_1[y\backslash\mathtt{C}_2[\Delta]] = [z\backslash v]\mathtt{L}'$ we have that $\mathtt{C}_2[\Delta] = v$. This is impossible since answers do not have redexes under non-answer evaluation contexts (Lem. 97).

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{11}$.** Then $\mathtt{C}_{11} = \mathtt{C}[\mathtt{C}']$, so by the decomposition lemma for evaluation contexts (Lem. 50) $\mathtt{C}$ is an evaluation context in $\mathcal{C}^h_{\vartheta'}$. Note that $\vartheta' = \mathsf{fz}^{\vartheta}([z\backslash v]\mathtt{L}') \subseteq \vartheta \cup \operatorname{dom}\mathtt{L}'$ and variables in $\operatorname{dom}\mathtt{L}$ cannot occur free in $\mathtt{C}$, since $\mathtt{C}$ is outside the scope of $\mathtt{L}$ in the original term t_0. In particular, the variables in $\operatorname{dom}\mathtt{L}$ do not occur as structural variables in $\mathtt{C}$, so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$. This in turn means that $\mathtt{C}[z\backslash v\mathtt{L}']$ is an evaluation context in $\mathcal{C}^h_{\vartheta}$, contradicting the fact that $\mathtt{r}$ is an internal step.

C. **The context $\mathtt{C}_{11}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{11}[\mathtt{C}']$. Given that $\mathtt{C}[v] = \mathtt{C}_{11}[\![y]\!]$ we have that $v = y$, which is impossible.

ii. **Case B.** Then $\mathtt{C}_1 = \mathtt{C}_{11}[\![x']\!]\mathscr{L}\{\Box\}$ such that:

$$\mathtt{C}_{11}[\![x']\!] = \mathtt{C}[v] \qquad \mathtt{L}_1 = \Box\mathscr{L}\{y\}$$

where $\vartheta' = \mathsf{fz}^{\vartheta}([z\backslash v]\mathtt{L}') = \mathsf{fz}^{\vartheta}(\mathtt{L}_1[y\backslash\mathtt{C}_1[\Delta]])$, the evaluation context $\mathtt{C}_{11}$ is in $\mathcal{C}^h_{\vartheta'}$, and $\mathscr{L}$ is a (ϑ, x')-chain context.

The remainder of this case is analogous to the previous item 5(a)i. For case 5(a)iA, recall that answers do not have variables under non-answer evaluation contexts (Lem. 97).

(b) **The internal step $\mathtt{r}$ is to the left of $t_0 = t_0'[y\backslash r_0]$.** Then there is a step $\mathtt{r}_1 : t_0' \to_{\mathtt{sh\backslash gc}} \mathtt{C}_1[\![y]\!]$. The step $\mathtt{r}_1$ must be ϑ-internal, for if it were ϑ-external, we would have that the step $\mathtt{r} : t_0'[y\backslash\mathtt{C}_2[\Delta]] \to_{\mathtt{sh\backslash gc}} \mathtt{C}_1[\![y]\!][y\backslash\mathtt{C}_2[\Delta]]$ is also ϑ-external, contradicting the hypothesis.

Since $\mathtt{r}_1$ is internal, by the fact that needed variables are backwards preserved by internal steps (Lem. 105) we have that t_0' is of the form $\mathtt{C}_0[\![y]\!]$, where $\mathtt{C}_0$ is an evaluation context in $\mathcal{C}^h_{\vartheta}$.

This allows us to close the diagram:

$$\begin{array}{ccc}
t_0 = \mathtt{C}_0[\![y]\!][y\backslash\mathtt{C}_2[\Delta]] & \xrightarrow{\neg\vartheta} & \mathtt{C}_1[\![y]\!][y\backslash\mathtt{C}_2[\Delta]] = t_1 \\
{\scriptstyle\vartheta}\downarrow & & \downarrow{\scriptstyle\vartheta} \\
t_2 = \mathtt{C}_0[\![y]\!][y\backslash\mathtt{C}_2[\Delta']] & \xrightarrow{\mathtt{sh\backslash gc}} & \mathtt{C}_1[\![y]\!][y\backslash\mathtt{C}_2[\Delta']] = t_3
\end{array}$$

(c) **The internal step $\mathtt{r}$ is to the right of $t_0 = t_0'[y\backslash r_0]$.** Then there is a step $\mathtt{r}_1 : r_0 \to_{\mathtt{sh\backslash gc}} \mathtt{C}_2[\Delta]$. We consider two subcases, depending on whether $\mathtt{r}_1$ is ϑ-external or ϑ-internal:

i. **If $\mathtt{r}_1$ is ϑ-external.** Then its source r_0 is of the form $r_0 = \mathtt{C}_3[\Sigma]$ where $\mathtt{C}_3 \in \mathcal{C}^h_{\vartheta}$ is an evaluation context and Σ is the anchor of a redex. Moreover, $h \neq \cdot$, since otherwise we would have that the step $\mathtt{r} : \mathtt{C}_1[\![y]\!][y\backslash r_0] \to_{\mathtt{sh\backslash gc}} \mathtt{C}_1[\![y]\!][y\backslash\mathtt{C}_2[\Delta']]$ is external, contradicting the hypothesis that it is internal.

So given that $h \neq \cdot$, then either h is a lambda or a constant and by Lem. 45 we conclude that r_0 is an answer. By the fact that answers are stable by reduction (Lem. 77) this means that $\mathtt{C}_2[\Delta']$ is also an answer, which contradicts the fact that answers do not have redexes below non-answer evaluation contexts (Lem. 97).

ii. **If $\mathtt{r}_1$ is ϑ-internal.** Then by *i.h.* we have that there exists a non-answer evaluation context $\mathtt{C}_3$, a $\mathtt{dB}$ redex Δ_0 and its contractum Δ_0' such that $r_0 = \mathtt{C}_3[\Delta_0]$ and:

$$\begin{tikzcd}
\mathtt{C}_3[\Delta_0] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_1[\Delta] \arrow[d, tail, "\vartheta"] \\
\mathtt{C}_3[\Delta_0'] \arrow[r, two heads, "\mathtt{sh\backslash gc}"] & \mathtt{C}_1[\Delta']
\end{tikzcd}$$

So by taking $\mathtt{C}_0 := \mathtt{C}_1[\![y]\!][y\backslash\mathtt{C}_3] \in \mathcal{C}^h_\vartheta$ we have:

$$\begin{tikzcd}
t_0 = \mathtt{C}_1[\![y]\!][y\backslash\mathtt{C}_3[\Delta_0]] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_1[\![y]\!][y\backslash\mathtt{C}_2[\Delta]] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = \mathtt{C}_1[\![y]\!][y\backslash\mathtt{C}_3[\Delta_0']] \arrow[r, two heads, "\mathtt{sh\backslash gc}"] & \mathtt{C}_1[\![y]\!][y\backslash\mathtt{C}_2[\Delta']] = t_3
\end{tikzcd}$$

6. **EAppRStr, $\mathtt{C} = r\,\mathtt{C}_1$, where $r \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ and $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$.** The situation is:

$$\begin{tikzcd}
t_0 \arrow[r, "\neg\vartheta"] & r\,\mathtt{C}_1[\Delta] = t_1 \arrow[d, tail, "\vartheta"] \\
& r\,\mathtt{C}_1[\Delta'] = t_3
\end{tikzcd}$$

There are three cases: (1) the step $\mathtt{r}$ is at the root of t_0; (2) t_0 is an application $r_0\,t_0'$ and the step takes place inside r_0; (3) t_0 is a substitution $r_0\,t_0'$ and the step takes place inside t_0'.

(a) **The internal step $\mathtt{r}$ is at the root of t_0.** This case is impossible. Note that $\mathtt{r}$ cannot be a $\mathtt{dB}$, $\mathtt{fix}$ or $\mathtt{case}$ step at the root, since it would be an external step. Moreover, $\mathtt{r}$ cannot be a $\mathtt{lsv}$ step at the root, since then the outermost constructor of $t_1 = r\,\mathtt{C}_1[\Delta]$ would be a substitution, but it is an application.

(b) **The internal step $\mathtt{r}$ is to the left of $t_0 = r_0\,t_0'$.** Then there is a step $\mathtt{r}_1 : r_0 \to_{\mathtt{sh\backslash gc}} r$. The step $\mathtt{r}_1$ cannot be ϑ-external, for this would imply that $\mathtt{r} : r_0\,\mathtt{C}_1[\Delta] \to_{\mathtt{sh\backslash gc}} r\,\mathtt{C}_1[\Delta]$ is also ϑ-external, contradicting the hypothesis.

Recall that $r \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$, so by the fact that normal forms are backwards preserved by internal steps (Lem. 104) we have that $r_0 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$. Hence we may close the diagram as follows:

$$\begin{tikzcd}
t_0 = r_0\,\mathtt{C}_1[\Delta] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & r\,\mathtt{C}_1[\Delta] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = r_0\,\mathtt{C}_1[\Delta'] \arrow[r, "\mathtt{sh\backslash gc}"] & r\,\mathtt{C}_1[\Delta'] = t_3
\end{tikzcd}$$

(c) **The internal step $\mathtt{r}$ is to the right of $t_0 = r_0\, t'_0$.** Then there is a step $\mathtt{r}_1 : t'_0 \to_{\mathtt{sh\backslash gc}} \mathtt{C}_1[\Delta]$. The step $\mathtt{r}_1$ cannot be ϑ-external, for this would imply that $\mathtt{r} : r_0\, \mathtt{C}_1[\Delta] \to_{\mathtt{sh\backslash gc}} r\, \mathtt{C}_1[\Delta]$ is also ϑ-external, contradicting the hypothesis.

By *i.h.* we have that there exists an evaluation context $\mathtt{C}_{10} \in \mathcal{C}^h_\vartheta$, a $\mathtt{dB}$ redex Δ_0 and Δ'_0 its contractum such that $t'_0 = \mathtt{C}_{10}[\Delta_0]$ and:

$$\begin{tikzcd}
\mathtt{C}_{10}[\Delta_0] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_1[\Delta] \arrow[d, tail, "\vartheta"] \\
\mathtt{C}_{10}[\Delta'_0] \arrow[r, two heads, "\mathtt{sh\backslash gc}"] & \mathtt{C}_1[\Delta']
\end{tikzcd}$$

So by taking $\mathtt{C}_0 := r\, \mathtt{C}_{10} \in \mathcal{C}^h_\vartheta$ we have:

$$\begin{tikzcd}
t_0 = r\, \mathtt{C}_{10}[\Delta_0] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & r\, \mathtt{C}_1[\Delta] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = r\, \mathtt{C}_{10}[\Delta'_0] \arrow[r, two heads, "\mathtt{sh\backslash gc}"] & r\, \mathtt{C}_1[\Delta'] = t_3
\end{tikzcd}$$

7. **EAppCons, $\mathtt{C} = r\, \mathtt{C}_1$, where $r \in \mathcal{K}_\vartheta$, $h = \mathsf{hc}(r)$, and $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$.** Similar to EAppRStr.

8. **ELam, $\mathtt{C} = \lambda y.\mathtt{C}$.** The internal step $\mathtt{r}$ is of the form:

$$\mathtt{r} : t_0 \to^{\neg\vartheta}_{\mathtt{sh}} \lambda y.\mathtt{C}[\Delta]$$

Note that $\mathtt{r}$ cannot be at the root of t_0, so t_0 must be an abstraction $\lambda y.t'_0$, and $\mathtt{r}$ must be internal to t'_0. Let $\mathtt{r}_1 : t'_0 \to_{\mathtt{sh\backslash gc}} \mathtt{C}[\Delta]$. Note that $\mathtt{r}_1$ cannot be a $(\vartheta \cup \{y\})$-external step, since this would imply that $\mathtt{r}$ is ϑ-external. So $\mathtt{r}_1$ is $(\vartheta \cup \{y\})$-internal. Then closing the diagram is straightforward by *i.h.*.

9. **ECase1.** Similar to previous cases.

10. **ECase2.** Similar to previous cases.

□

Lemma 107 (Permutation of internal steps and external $\mathtt{lsv}$ steps)**.** *Given any set of variables ϑ such that $\mathsf{fv}(t_0) \subseteq \vartheta$, if $t_0 \to^{\neg\vartheta}_{\mathtt{sh}} t_1 \rightarrowtail^{\vartheta}_{\mathtt{sh}} t_3$ where the second step is a $\mathtt{lsv}$ step, there exists a term t_2 such that $t_0 \rightarrowtail^{\vartheta}_{\mathtt{sh}} t_2 \twoheadrightarrow_{\mathtt{sh\backslash gc}} t_3$ where the first step is a $\mathtt{lsv}$ step. An explicit construction for the diagrams is given.*

Proof. Let $\mathtt{r}$ be the internal step $t_0 \to^{\neg\vartheta}_{\mathtt{sh}} t_1$ and $\mathtt{r}'$ the external $\mathtt{lsv}$ step $t_1 \rightarrowtail^{\vartheta}_{\mathtt{sh}} t_3$. Then $t_1 = \mathtt{C}_1[\mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}]]$ and $t_3 = \mathtt{C}_1[\mathtt{C}_2[v][x\backslash v]\mathtt{L}]$, where $\mathtt{C}_1[\mathtt{C}_2[x\backslash v\mathtt{L}]] \in \mathcal{C}^h_\vartheta$. We write Δ to stand for the $\mathtt{lsv}$ redex $\mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}]$ and Δ' for its contractum $\mathtt{C}_2[v][x\backslash v]\mathtt{L}$.

By induction on the derivation that $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$, the term t_0 will be shown to be of the form $\mathtt{C}_{10}[\mathtt{C}_{20}[\![x]\!][x\backslash v_0\mathtt{L}_0]]$, where $\mathtt{C}_{10}[\mathtt{C}_{20}[x\backslash v_0\mathtt{L}_0]] \in \mathcal{C}^h_\vartheta$, and then $t_2 = \mathtt{C}_{10}[\mathtt{C}_{20}[v_0][x\backslash v_0]\mathtt{L}_0]$, in such a way that the diagram is closed as required by the statement. We write Δ_0 to stand for the $\mathtt{lsv}$ redex $\mathtt{C}_{20}[\![x]\!][x\backslash v_0\mathtt{L}_0]$ and Δ'_0 for its contractum $\mathtt{C}_{20}[v][x\backslash v_0]\mathtt{L}_0$.

Furthermore, suppose that $\mathtt{C}_2 \in \mathcal{C}^{h'}_{\vartheta'}$. Then the inductive construction will ensure that $\mathtt{C}_{20} \in \mathcal{C}^{h'}_{\vartheta''}$.

1. **EBox,** $\mathtt{C}_1 = \Box \in \mathcal{C}^h_\vartheta$**.** Note that in this case $\vartheta' = \mathsf{fz}^\vartheta(\Box) = \vartheta$. Then there is a $\mathtt{lsv}$ redex at the root of $t_1 = \mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}]$. We consider three cases: (1) the step $\mathtt{r} : t_0 \to^{\neg\vartheta}_{\mathtt{sh}} t_1$ is at the root of t_0; (2) t_0 is a substitution $t'_0[x\backslash s_0]$ and $\mathtt{r}$ is internal to t'_0; (3) t_0 is a substitution $t'_0[x\backslash s_0]$ and $\mathtt{r}$ is internal to s_0.

   (a) **The internal step $\mathtt{r}$ is at the root of t_0.** Note that $\mathtt{r}$ cannot be a $\mathtt{dB}$, $\mathtt{fix}$ or $\mathtt{case}$ step, since it would be external. So $\mathtt{r}$ is a $\mathtt{lsv}$ step, *i.e.*:

   $$t_0 = \mathtt{C}[\![y]\!][y\backslash v'\mathtt{L}'] \to^{\neg\vartheta}_{\mathtt{sh}} \mathtt{C}[v'][y\backslash v']\mathtt{L}' = t_1 = \mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}]$$

   Let $\mathtt{L}_1$ be a substitution context such that $[y\backslash v']\mathtt{L}' = \mathtt{L}_1[x\backslash v\mathtt{L}]$, and using Lem. 95 let us strip $\mathtt{L}_1$ from $\mathtt{C}_2[\![x]\!]$. This gives us two possibilities, case **A** and case **B** in the statement of Lem. 95:

   i. **Case A.** Then $\mathtt{C}_2[\![x]\!] = \mathtt{C}_{21}[\![x]\!]\mathtt{L}_1$ where $\vartheta'' = \mathsf{fz}^\vartheta(\mathtt{L}_1)$, the evaluation context $\mathtt{C}_{21}$ is in $\mathcal{C}^h_{\vartheta''}$ and we have:

   $$\mathtt{C}[v'] = \mathtt{C}_{21}[\![x]\!]$$

   We consider three further subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{21}$.

   A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{21}$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:

   $$\widehat{\mathtt{C}}[\Box, v] = \mathtt{C}_{21} \qquad \widehat{\mathtt{C}}[x, \Box] = \mathtt{C}$$

   By the fact that evaluation contexts are backwards-stable by substitutions (Lem. 89) there are two possibilities for $\widehat{\mathtt{C}}$: the left and the right branch of the disjunction. Let us analyze each branch:

   - **Left branch.** Then $\widehat{\mathtt{C}}[\Box, y]$ is an evaluation context in $\mathcal{C}^h_{\vartheta''}$. Note that $\vartheta'' = \mathsf{fz}^\vartheta(\mathtt{L}_1[x\backslash v\mathtt{L}]) = \mathsf{fz}^\vartheta([y\backslash v']\mathtt{L}') \subseteq \vartheta \cup \operatorname{dom}\mathtt{L}'$, and that variables in $\operatorname{dom}\mathtt{L}'$ do not occur in $\widehat{\mathtt{C}}[\Box, y]$, since $\widehat{\mathtt{C}}[\Box, y]$ is outside the scope of $\mathtt{L}'$ in the original term t_0. By repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we obtain that $\widehat{\mathtt{C}}[\Box, y] \in \mathcal{C}^h_\vartheta$.
     Note also that x is bound by $\mathtt{L}_1[x\backslash v\mathtt{L}] = [y\backslash v']\mathtt{L}'$, and that it must occur bound in $t_0 = \widehat{\mathtt{C}}[x, y][y\backslash v'\mathtt{L}']$, since free variables cannot become bound. So x it must be bound by $[y\backslash v'\mathtt{L}']$, which means that $x = y$, and in particular $v = v'$ and $\mathtt{L} = \mathtt{L}'$. We may then close the diagram as follows:

     $$\begin{tikzcd}
     t_0 = \widehat{\mathtt{C}}[x, x][x\backslash v\mathtt{L}] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \widehat{\mathtt{C}}[x, v][x\backslash v]\mathtt{L} = t_1 \arrow[d, tail, "\vartheta"] \\
     t_2 = \widehat{\mathtt{C}}[v, x][x\backslash v]\mathtt{L} \arrow[r, "\mathtt{sh}\backslash\mathtt{gc}"] & \widehat{\mathtt{C}}[v, v][x\backslash v]\mathtt{L} = t_3
     \end{tikzcd}$$

   - **Right branch.** Then $\mathtt{C} = \widehat{\mathtt{C}}[x, \Box]$ is an evaluation context in $\mathcal{C}^h_{\vartheta''}$. Note that $\vartheta'' = \mathsf{fz}^\vartheta([x\backslash v']\mathtt{L}') \subseteq \vartheta \cup \operatorname{dom}\mathtt{L}'$ and variables in $\operatorname{dom}\mathtt{L}'$ do not occur in $\mathtt{C}$, since $\mathtt{C}$ is outside the scope of $\mathtt{L}$ in the original term t_0. By repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we obtain that $\mathtt{C} \in \mathcal{C}^h_\vartheta$. We conclude that the step $\mathtt{r} : \mathtt{C}[\![y]\!][y\backslash v'\mathtt{L}'] \to^{\neg\vartheta}_{\mathtt{sh}} \mathtt{C}[v'][y\backslash v']\mathtt{L}'$ is external, which contradicts the hypothesis that it is internal.

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{21}$.** Then $\mathtt{C}_{21} = \mathtt{C}[\mathtt{C}']$. By the decomposition of evaluation contexts lemma (Lem. 50) we know that $\mathtt{C}$ must be an evaluation context in $\mathbb{X}_{\vartheta''}$. By the fact that non-structural variables are not required in "ϑ" (Lem. 87) we obtain that $\mathtt{C} \in \mathcal{C}^h_\vartheta$. This contradicts the hypothesis that $\mathtt{r}$ is an internal step.

C. **The context $\mathtt{C}_{21}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{21}[\mathtt{C}']$. Since $\mathtt{C}[v'] = \mathtt{C}_{21}\llbracket x\rrbracket$ this implies that $v' = x$, which is impossible.

ii. **Case B.** Then $\mathtt{C}_2\llbracket x\rrbracket = \mathtt{C}_{21}\llbracket z\rrbracket\mathscr{L}\{x\}$ such that:

$$\mathtt{C}[v'] = \mathtt{C}_{21}\llbracket z\rrbracket \qquad \Box\mathscr{L}\{x\} = \mathtt{L}_1$$

where $\vartheta'' = \mathsf{fz}^\vartheta(\mathtt{L}_1)$, the evaluation context $\mathtt{C}_{21}$ is in $\mathcal{C}^h_{\vartheta''}$, and $\mathscr{L}$ is a (ϑ, z)-chain context.

The remainder of this case is by case analysis on the relative positions of the hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{21}$, similar to item 1(a)i. The only significant difference is for 1(a)iA in the **Left branch** subcase. In this subcase we have that:

$$t_0 = \widehat{\mathtt{C}}[z, y][y\backslash v'\mathtt{L}'] \rightarrow^{\neg\vartheta}_{\mathtt{sh}} \widehat{\mathtt{C}}[z, v'][y\backslash v']\mathtt{L}' = \widehat{\mathtt{C}}[z, v']\mathscr{L}\{x\}[x\backslash v\mathtt{L}] = t_1$$

where $\widehat{\mathtt{C}}[\Box, y] \in \mathbb{X}_\vartheta$ and $\mathscr{L}$ is a (ϑ, z)-chain context. Note that z must be bound by $[y\backslash v']\mathtt{L}'$, so $z = y$. Moreover, z must also be bound by $\Box\mathscr{L}\{x\}$ to a term of the form $\mathtt{I}^{\hat\vartheta}\llbracket x'\rrbracket$. Thus $v' = \mathtt{I}^{\hat\vartheta}\llbracket x'\rrbracket$ which is a contradiction, since answers do not have occurrences of variables below non-answer evaluation contexts (Lem. 97).

(b) **The internal step $\mathtt{r}$ is to the left of $t_0 = t'_0[x\backslash s_0]$.** Then there is a step $\mathtt{r}_1 : t'_0 \rightarrow_{\mathtt{sh\backslash gc}} \mathtt{C}_2\llbracket x\rrbracket$. Note that $\mathtt{r}_1$ cannot be ϑ-external, for this would imply that $\mathtt{r}$ is ϑ-external. Hence $\mathtt{r}_1$ is ϑ-internal, so by the fact that needed variables are backwards preserved by internal steps (Lem. 105) we have that there is an evaluation context $\mathtt{C}_{20} \in \mathcal{C}^h_\vartheta$ such that $t'_0 = \mathtt{C}_{20}\llbracket x\rrbracket$. Thus the diagram can be closed as follows:

$$\begin{tikzcd}
t_0 = \mathtt{C}_{20}\llbracket x\rrbracket[x\backslash v\mathtt{L}] \arrow[r, "\neg\vartheta"] \arrow[d, "\vartheta"'] & \mathtt{C}_2\llbracket x\rrbracket[x\backslash v\mathtt{L}] = t_1 \arrow[d, "\vartheta"] \\
t_2 = \mathtt{C}_{20}[v][x\backslash v]\mathtt{L} \arrow[r, "\mathtt{sh\backslash gc}"] & \mathtt{C}_2[v][x\backslash v]\mathtt{L} = t_3
\end{tikzcd}$$

(c) **The internal step $\mathtt{r}$ is to the right of $t_0 = t'_0[x\backslash s_0]$.** Then there is a step $\mathtt{r}_1 : s_0 \rightarrow_{\mathtt{sh\backslash gc}} v\mathtt{L}$. We consider two cases, depending on whether $\mathtt{r}_1$ is a ϑ-external or a ϑ-internal step:

i. **If $\mathtt{r}_1$ is ϑ-external.** Then s_0 is of the form $\mathtt{C}_3[\Sigma]$, where $\mathtt{C}_3$ is an evaluation context in $\mathcal{C}^{h'}_\vartheta$ and Σ is the anchor of a redex. Note that $h \neq \cdot$ since otherwise the context $\mathtt{C}_2\llbracket x\rrbracket[x\backslash \mathtt{C}_3]$ would be an evaluation context, and the step:

$$\mathtt{r} : \mathtt{C}_2\llbracket x\rrbracket[x\backslash \mathtt{C}_3[\Sigma]] \rightarrow_{\mathtt{sh\backslash gc}} \mathtt{C}_2\llbracket x\rrbracket[x\backslash v\mathtt{L}]$$

would be external, contradicting the hypothesis that it is internal.

Then since $h \neq \cdot$ by the Lem. 45 we may conclude that $\mathtt{C}_3[\Sigma] = v_0\mathtt{L}_0$. Hence the diagram can be closed as follows:

$$\begin{tikzcd}
t_0 = \mathtt{C}_2[\![x]\!][x\backslash v_0\mathtt{L}_0] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = \mathtt{C}_2[v_0][x\backslash v_0]\mathtt{L}_0 \arrow[r, "\mathtt{sh}\backslash\mathtt{gc}"] & \mathtt{C}_2[v][x\backslash v]\mathtt{L} = t_3
\end{tikzcd}$$

ii. **If $\mathtt{r}_1$ is ϑ-internal.** Then by the fact that answers are backwards stable by internal steps (Lem. 78) we have that s_0 is of the form $s_0 = v_0\mathtt{L}_0$, and the diagram can be closed just as in the previous case.

2. **EAppL, $\mathtt{C}_1 = \mathtt{C}_3\, t$ and $h \neq \lambda$ and $\mathtt{C}_3 \in \mathcal{C}^h_\vartheta$.** The situation is:

$$\begin{tikzcd}
t_0 \arrow[r, "\neg\vartheta"] & \mathtt{C}_3[\Delta]\, t = t_1 \arrow[d, tail, "\vartheta"] \\
& \mathtt{C}_3[\Delta']\, t = t_3
\end{tikzcd}$$

where $\mathtt{C}_3[\mathtt{C}_2[x\backslash v\mathtt{L}]] \in \mathcal{C}^{h'}_\vartheta$ with $h' \neq \lambda$.

This case is analogous to item 2 of the previous lemma (Lem. 106), as the proof does not rely on Δ being a $\mathtt{dB}$ redex.

3. **ESubLNonStr, $\mathtt{C}_1 = \mathtt{C}_{11}[y\backslash t]$, where $y \notin \vartheta$, $t \notin \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$, and $\mathtt{C}_{11} \in \mathcal{C}^h_\vartheta$.** The situation is:

$$\begin{tikzcd}
t_0 \arrow[r, "\neg\vartheta"] & \mathtt{C}_{11}[\Delta][y\backslash t] = t_1 \arrow[d, tail, "\vartheta"] \\
& \mathtt{C}_{11}[\Delta'][y\backslash t] = t_3
\end{tikzcd}$$

We consider three cases, depending on whether (1) the internal step $\mathtt{r}$ is at the root of t_0, (2) t_0 is a substitution $t'_0[y\backslash r_0]$ and the step $\mathtt{r}$ is internal to t'_0, (3) t_0 is a substitution $t'_0[y\backslash r_0]$ and the step $\mathtt{r}$ is internal to r_0.

(a) **The internal step $\mathtt{r}$ is at the root of t_0.** Then $\mathtt{r}$ cannot be a $\mathtt{dB}$, $\mathtt{fix}$ or $\mathtt{case}$ step, since it would be external. So it must be a $\mathtt{lsv}$ step. Then the step $\mathtt{r}$ is of the form:

$$t_0 = \mathtt{C}[\![z]\!][z\backslash v'\mathtt{L}'] \to^{\neg\vartheta}_{\mathtt{sh}} \mathtt{C}[v'][z\backslash v']\mathtt{L}' = \mathtt{C}_{11}[\Delta][y\backslash t] = t_1$$

Let $\mathtt{L}_1$ be a substitution context such that $\mathtt{L}_1[y\backslash t] = [z\backslash v']\mathtt{L}'$. Recall that $\Delta = \mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}]$. Using Lem. 96 let us strip $\mathtt{L}_1$ from $\mathtt{C}_{11}[\mathtt{C}_2[x\backslash v\mathtt{L}]]$. This gives us four possibilities, **A**, **B**, **C**, and **D** in the statement of Lem. 96.

i. **Case A.** Then:

$$\mathtt{C}_{11} = \mathtt{C}_{111}\mathtt{L}_1 \qquad \mathtt{C}[v'] = \mathtt{C}_{111}[\Delta]$$

where $\hat{\vartheta} = \mathsf{fz}^\vartheta(\mathtt{L}_1)$ and $\mathtt{C}_{111} \in \mathcal{C}^{h'}_{\hat{\vartheta}}$.
We consider three further subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{111}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{111}$ are disjoint.** Then there is a two hole context $\widehat{\mathtt{C}}$ such that

$$\widehat{\mathtt{C}}[\Box, v'] = \mathtt{C}_{111} \qquad \widehat{\mathtt{C}}[\Delta, \Box] = \mathtt{C}$$

By the fact that evaluation contexts are backwards-stable by substitutions (Lem. 89) there are two possibilities for $\widehat{\mathtt{C}}$: the left and the right branch of the disjunction. Let us analyze each branch:

- **Left branch.** Then $\widehat{\mathtt{C}}[\Box, z]$ is an evaluation context in $\mathcal{C}^{h'}_{\hat{\vartheta}}$. Since $\hat{\vartheta} = \mathtt{fz}^{\vartheta}([z\backslash v']\mathtt{L}') \subseteq \vartheta \cup \mathrm{dom}\,\mathtt{L}'$, by applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we obtain that $\widehat{\mathtt{C}}[\Box, z] \in \mathcal{C}^{h'}_{\vartheta}$. This allows us to close the diagram as follows:

$$\begin{array}{ccc} t_0 = \widehat{\mathtt{C}}[\Delta, z][z\backslash v'\mathtt{L}'] & \xrightarrow{\neg\vartheta} & \widehat{\mathtt{C}}[\Delta, v'][z\backslash v']\mathtt{L}' = t_1 \\ \vartheta\downarrow & & \downarrow\vartheta \\ t_2 = \widehat{\mathtt{C}}[\Delta', z][z\backslash v'\mathtt{L}'] & \xrightarrow{\mathtt{sh}\backslash\mathtt{gc}} & \widehat{\mathtt{C}}[\Delta', v'][z\backslash v']\mathtt{L}' = t_3 \end{array}$$

- **Right branch.** Then $\mathtt{C} \in \mathcal{C}^{h'}_{\hat{\vartheta}}$. Since $\hat{\vartheta} \subseteq \vartheta \cup \mathrm{dom}\,\mathtt{L}'$, by applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we obtain that $\mathtt{C} \in \mathcal{C}^{h'}_{\vartheta}$. This contradicts the fact that $\mathtt{r}$ is an internal step.

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{111}$.** Then by the decomposition of evaluation contexts lemma (Lem. 50) we know that $\mathtt{C} \in \mathcal{C}^{h'}_{\hat{\vartheta}}$. Since $\hat{\vartheta} \subseteq \vartheta \cup \mathrm{dom}\,\mathtt{L}'$, by applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we obtain that $\mathtt{C} \in \mathcal{C}^{h'}_{\vartheta}$. This contradicts the fact that $\mathtt{r}$ is an internal step.

C. **The context $\mathtt{C}_{111}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{111}[\mathtt{C}_1]$, so $\mathtt{C}_1[v'] = \mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}]$. We proceed by case analysis on the position of the hole of $\mathtt{C}_1$ in the term $\mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}]$: it can be to the left of the substitution $[x\backslash v\mathtt{L}]$, or inside the substitution.

- **Left of the substitution, $\mathtt{C}_1 = \mathtt{C}_{11}[x\backslash v\mathtt{L}]$.** Now $\mathtt{C}_{11}[v'] = \mathtt{C}_2[\![x]\!]$. Let us analyze the relative positions of the holes of the contexts $\mathtt{C}_{11}$ and $\mathtt{C}_2$. Observe that $\mathtt{C}_2$ cannot be a prefix of $\mathtt{C}_{11}$, as this would imply that $x = \mathtt{C}_2[v]$. So there are two possibilities, either the holes of $\mathtt{C}_{11}$ and $\mathtt{C}_2$ are disjoint, or $\mathtt{C}_{11}$ is a prefix of $\mathtt{C}_2$:
  - **If the holes of $\mathtt{C}_{11}$ and $\mathtt{C}_2$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:

  $$\widehat{\mathtt{C}}[\Box, v'] = \mathtt{C}_2 \qquad \widehat{\mathtt{C}}[x, \Box] = \mathtt{C}_{11}$$

  By the fact that evaluation contexts are backwards-stable by substitutions (Lem. 89) there are two possibilities for $\widehat{\mathtt{C}}$: the left and the right branch of the disjunction. Let us analyze each branch:
  Left branch Then $\widehat{\mathtt{C}}[\Box, z] \in Y^{\vartheta'}$, so we may close the diagram as follows:

$$\begin{array}{ccc} t_0 = \mathtt{C}_{111}[\widehat{\mathtt{C}}[x, z][x\backslash v\mathtt{L}]][z\backslash v'\mathtt{L}'] & \xrightarrow{\neg\vartheta} & \mathtt{C}_{111}[\widehat{\mathtt{C}}[x, v'][x\backslash v\mathtt{L}]][z\backslash v']\mathtt{L}' = t_1 \\ \vartheta\downarrow & & \downarrow\vartheta \\ t_2 = \mathtt{C}_{111}[\widehat{\mathtt{C}}[v, z][x\backslash v]\mathtt{L}][z\backslash v'\mathtt{L}'] & \xrightarrow{\mathtt{sh}\backslash\mathtt{gc}} & \mathtt{C}_{111}[\widehat{\mathtt{C}}[v, v'][x\backslash v]\mathtt{L}][z\backslash v']\mathtt{L}' = t_3 \end{array}$$

Right branch Then $\mathtt{C}_{11} \in Y$. Hence the context $\mathtt{C} = \mathtt{C}_{111}[\mathtt{C}_{11}[x\backslash v\mathtt{L}]][z\backslash v'\mathtt{L}']$ is in $\mathcal{C}^h_\vartheta$. This contradicts the fact that $\mathtt{r}$ is internal.

– **If $\mathtt{C}_{11}$ is a prefix of $\mathtt{C}_2$.** Then $\mathtt{C}_2 = \mathtt{C}_{11}[\mathtt{C}_2]$. The situation is:

$$\begin{array}{rl} & t_0 = \mathtt{C}_{111}[\mathtt{C}_{11}[\![z]\!][x\backslash v\mathtt{L}]][z\backslash v'\mathtt{L}'] \\ \to^{\neg\vartheta}_{\mathtt{sh}} & \mathtt{C}_{111}[\mathtt{C}_{11}[v'][x\backslash v\mathtt{L}]][z\backslash v']\mathtt{L}' = t_1 \end{array}$$

and we have that $\mathtt{C}_2[\![x]\!] = \mathtt{C}_{11}[v']$. Given that $\mathtt{C}_{11}$ is a prefix of $\mathtt{C}_2$, we have in particular that x occurs free in v'. This is impossible by Barendregt's variable convention, since v' is outside the scope of the substitution binding x in t_0.

- **Inside the substitution, $\mathtt{C}_1 = \mathtt{C}_2[\![x]\!][x\backslash\mathtt{C}_{11}]$.** So $\mathtt{C}_1[v'] = v\mathtt{L}$. We consider two further subcases, depending on whether the hole of $\mathtt{C}_1$ is inside v or inside one of the substitutions in $\mathtt{L}$.

– **If $\mathtt{C}_1 = \mathtt{C}_{111}\mathtt{L}$ and $v = \mathtt{C}_{111}[v']$.** There are two possibilities, depending on whether the context $\mathtt{C}_{111}$ is empty:
Empty, *i.e.* $\mathtt{C}_{111} = \Box$ Then the situation is:

$$\begin{array}{rl} & t_0 = \mathtt{C}_{111}[\mathtt{C}_2[\![x]\!][x\backslash z\mathtt{L}]][z\backslash v'\mathtt{L}'] \\ \to^{\neg\vartheta}_{\mathtt{sh}} & \mathtt{C}_{111}[\mathtt{C}_2[\![x]\!][x\backslash v'\mathtt{L}]][z\backslash v']\mathtt{L}' = t_1 \end{array}$$

Note that the context $\mathtt{C}_{111}[\mathtt{C}_2[\![x]\!][x\backslash\Box\mathtt{L}]][z\backslash v'\mathtt{L}']$ is a ϑ-evaluation context, so the step $\mathtt{r}$ is external, contradicting the hypothesis that it is internal.
Non-empty, *i.e.* $\mathtt{C}_{111} = \lambda x'.\mathtt{C}_2$ Then if we let $v_\Phi = \lambda x'.\mathtt{C}_2[\Phi]$ the diagram can be closed as follows:

$$\begin{tikzcd} t_0 = \mathtt{C}_{111}[\mathtt{C}_2[\![x]\!][x\backslash v_z\mathtt{L}]][z\backslash v'\mathtt{L}'] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_{111}[\mathtt{C}_2[\![x]\!][x\backslash v_{v'}\mathtt{L}]][z\backslash v']\mathtt{L}' = t_1 \arrow[d, tail, "\vartheta"] \\ t_2 = \mathtt{C}_{111}[\mathtt{C}_2[v_z][x\backslash v_z]\mathtt{L}][z\backslash v'\mathtt{L}'] \arrow[r, two heads, "\mathtt{sh}\backslash\mathtt{gc}"] & \mathtt{C}_{111}[\mathtt{C}_2[v_{v'}][x\backslash v_{v'}]\mathtt{L}][z\backslash v']\mathtt{L}' = t_3 \end{tikzcd}$$

– **If $\mathtt{C}_1 = v\mathtt{L}_1[y\backslash\mathtt{C}_{111}]\mathtt{L}_2$ and $\mathtt{L} = \mathtt{L}_1[y\backslash\mathtt{C}_{111}[v']]\mathtt{L}_2$.** Then if we let $\mathtt{L}_\Phi = \mathtt{L}_1[y\backslash\mathtt{C}_{111}[\Phi]]\mathtt{L}_2$ the diagram can be closed as follows:

$$\begin{tikzcd} t_0 = \mathtt{C}_{111}[\mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}_z]][z\backslash v'\mathtt{L}'] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_{111}[\mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}_{v'}]][z\backslash v']\mathtt{L}' = t_1 \arrow[d, tail, "\vartheta"] \\ t_2 = \mathtt{C}_{111}[\mathtt{C}_2[v][x\backslash v]\mathtt{L}_z][z\backslash v'\mathtt{L}'] \arrow[r, "\mathtt{sh}\backslash\mathtt{gc}"] & \mathtt{C}_{111}[\mathtt{C}_2[v][x\backslash v]\mathtt{L}_{v'}][z\backslash v']\mathtt{L}' = t_3 \end{tikzcd}$$

ii. **Case B.** Then:

$$\mathtt{C}_{11} = \mathtt{C}_{111}[\![w]\!]\mathscr{L}\{\Box\} \qquad \mathtt{C}[v'] = \mathtt{C}_{111}[\![w]\!] \qquad \mathtt{L}_1 = \Box\mathscr{L}\{\Delta\}$$

where $\hat{\vartheta} = \mathsf{fz}^\vartheta(\mathtt{L}_1)$, the evaluation context $\mathtt{C}_{111}$ is in $\mathbb{X}_{\hat{\vartheta}}$, and $\mathscr{L}$ is a (ϑ, w)-chain context.
We consider three further subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{111}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{111}$ are disjoint.** Then there is a two hole context $\widehat{\mathtt{C}}$ such that

$$\widehat{\mathtt{C}}[\Box, v'] = \mathtt{C}_{111} \qquad \widehat{\mathtt{C}}[w, \Box] = \mathtt{C}$$

By the fact that evaluation contexts are backwards-stable by substitutions (Lem. 89) there are two possibilities for $\widehat{\mathtt{C}}$: the left and the right branch of the disjunction. Let us analyze each branch:

- **Left branch.** Then $\widehat{\mathtt{C}}[\Box, z] \in \mathbb{X}_{\hat{\vartheta}}$. Note that in the term t_1, the variable w is bound by $\Box\mathscr{L}\{\Delta\}[y\backslash t] = [z\backslash v']\mathtt{L}'$ since $\mathscr{L}$ is a (ϑ, w)-chain context. Then w must also occur bound in the term $t_0 = \widehat{\mathtt{C}}[w, v'][z\backslash v'\mathtt{L}']$, since reduction cannot make a free variable become bound. Hence $w = z$.
  Consider the binding of w in the substitution context $\Box\mathscr{L}\{\Delta\}$. We know that it is of the form $\mathtt{I}^{\vartheta_1}[\Sigma]$ where $\mathtt{I}^{\vartheta_1}$ is a non-answer evaluation context for some value of ϑ_1, and Σ is either Δ (if $\mathscr{L}$ has exactly one jump) or a variable (if $\mathscr{L}$ has more than one jump). So we have that $v'\mathtt{L}' = \mathtt{I}^{\vartheta_1}[\Sigma]$. This is impossible by the fact that answers do not have redexes or variables below non-answer evaluation contexts (Lem. 97).
- **Right branch.** Then $\mathtt{C} \in \mathbb{X}_{\hat{\vartheta}}$. Since $\hat{\vartheta} \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}'$, by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$. This contradicts the hypothesis that the step $\mathtt{r}$ is internal.

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{111}$.** Then by the decomposition of evaluation contexts lemma (Lem. 50) we know that $\mathtt{C}_1 \in \mathbb{X}_{\hat{\vartheta}}$. Since $\hat{\vartheta} \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}'$, by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$. This contradicts the hypothesis that the step $\mathtt{r}$ is internal.

C. **The context $\mathtt{C}_{111}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{111}[\mathtt{C}_1]$, so $w = \mathtt{C}_1[\![v']\!]$, which is impossible.

iii. **Case C.** Then $\mathtt{C}_{11}$ is a substitution context, and:

$$\mathtt{C}_2 = \mathtt{C}_{21}\mathtt{L}_2 \qquad \mathtt{L}_1 = \mathtt{C}_{11}[\mathtt{L}_2[x\backslash v\mathtt{L}]] \qquad \mathtt{C}[v'] = \mathtt{C}_{21}[\![x]\!]$$

where $\hat{\vartheta}' = \mathsf{fz}^{\vartheta'}(\mathtt{L}_2)$ and the evaluation context $\mathtt{C}_{21}$ is in $Y^{\hat{\vartheta}'}$.
We consider three further subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{21}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{21}$ are disjoint.** Then there is a two hole context $\widehat{\mathtt{C}}$ such that

$$\widehat{\mathtt{C}}[\Box, v'] = \mathtt{C}_{21} \qquad \widehat{\mathtt{C}}[x, \Box] = \mathtt{C}$$

By the fact that evaluation contexts are backwards-stable by substitutions (Lem. 89) there are two possibilities for $\widehat{\mathtt{C}}$: the left and the right branch of the disjunction. Let us analyze each branch:

- **Left branch.** Then $\widehat{\mathtt{C}}[\Box, z] \in Y^{\hat{\vartheta}'}$. The step $\mathtt{r}$ is of the form:

$$\begin{array}{rl} & t_0 = \widehat{\mathtt{C}}[x, z][z\backslash \mathtt{C}_{11}[v'\mathtt{L}_2[x\backslash v\mathtt{L}]]][y\backslash r] \\ \to^{\neg\vartheta}_{\mathtt{sh}} & \mathtt{C}_{11}[\widehat{\mathtt{C}}[x, v'][z\backslash v']\mathtt{L}_2[x\backslash v\mathtt{L}]][y\backslash r] = t_1 \end{array}$$

  This is impossible by Barendregt's variable convention, since the variable x occurs free in t_0 and becomes bound to the substitution $[x\backslash v\mathtt{L}]$ in t_1.

- **Right branch.** Then $\mathtt{C} \in Y^{\hat{\vartheta}'}$. Since $\hat{\vartheta}' \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}'$ by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C} \in Y^{\vartheta}$. This contradicts the hypothesis that $\mathtt{r}$ is internal.

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{21}$.** Then by the decomposition lemma for evaluation contexts (Lem. 50) we know that $\mathtt{C} \in Y^{\hat{\vartheta}'}$. Since $\hat{\vartheta}' \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}'$ by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C} \in Y^{\vartheta}$. This contradicts the hypothesis that $\mathtt{r}$ is internal.

C. **The context $\mathtt{C}_{21}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{21}[\mathtt{C}_1]$ and in particular $x = \mathtt{C}_1[v']$, which is impossible.

iv. **Case D.** Then $\mathtt{C}_{11}$ is a substitution context, and:

$$\mathtt{C}_2 = \mathtt{C}_{21}[\![w]\!]\mathscr{L}\{\Box\} \qquad \mathtt{L}_1 = \mathtt{C}_{11}[\Box\mathscr{L}\{x\}[x\backslash v\mathtt{L}]]$$
$$\mathtt{C}[v'] = \mathtt{C}_{21}[\![w]\!]$$

where $\hat{\vartheta}' = \mathtt{fz}^{\vartheta'}(\Box\mathscr{L}\{x\})$, the evaluation context $\mathtt{C}_{21}$ is in $Y^{\hat{\vartheta}'}$, and $\mathscr{L}$ is a (ϑ', w)-chain context.

We consider three further subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{21}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{21}$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\Box, v'] = \mathtt{C}_{21} \qquad \widehat{\mathtt{C}}[w, \Box] = \mathtt{C}$$

By the fact that evaluation contexts are backwards-stable by substitutions (Lem. 89) there are two possibilities for $\widehat{\mathtt{C}}$: the left and the right branch of the disjunction. Let us analyze each branch:

- **Left branch.** Then $\widehat{\mathtt{C}}[\Box, z] \in Y^{\hat{\vartheta}'}$. Note that w is bound on the term $t_1 = \mathtt{C}_{11}[\widehat{\mathtt{C}}[w, v']\mathscr{L}\{x\}[x\backslash v\mathtt{L}]][y\backslash r]$, since $\mathscr{L}$ is a (ϑ', w)-chain context. So it must also be bound on the term $t_0 = \widehat{\mathtt{C}}[w, z][z\backslash v'\mathtt{L}']$, which means that $w = z$. In particular, consider the binding of w in $\Box\mathscr{L}\{x\}$. It is of the form $\mathtt{I}^{\vartheta_1}[\![x']\!]$ for some set ϑ_1 and some variable x'. Moreover, since $w = z$, it is also bound to v'. Hence $v' = \mathtt{I}^{\vartheta_1}[\![x']\!]$. This is impossible since answers do not have variables under non-answer contexts (Lem. 97).
- **Right branch.** Then $\mathtt{C} \in Y^{\hat{\vartheta}'}$. Since $\vartheta' \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}'$, by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C} \in Y^{\vartheta}$, contradicting that $\mathtt{r}$ is internal.

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{21}$.** Then by the decomposition lemma for evaluation contexts (Lem. 50) we know that $\mathtt{C} \in Y^{\hat{\vartheta}'}$. Since $\vartheta' \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}'$, by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C} \in Y^{\vartheta}$, contradicting that $\mathtt{r}$ is internal.

C. **The context $\mathtt{C}_{21}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{21}[\mathtt{C}']$. In particular $w = \mathtt{C}'[v]$, which is impossible.

(b) **The internal step $\mathtt{r}$ is to the left of $t_0 = t_0'[y\backslash r_0]$.** Then there is a step $\mathtt{r}_1 : t_0' \to_{\mathtt{sh}\backslash\mathtt{gc}} \mathtt{C}_{11}[\Delta]$. Note that $\mathtt{r}_1$ must be ϑ-internal, for otherwise $\mathtt{r}$ would be ϑ-external.

By *i.h.* we have that there exists an evaluation context $\mathtt{C}_{110} \in \mathcal{C}^h_\vartheta$, a $\mathtt{lsv}$ redex Δ_0 and Δ'_0 its contractum such that $t'_0 = \mathtt{C}_{110}[\Delta_0]$ and:

$$\begin{tikzcd}
\mathtt{C}_{110}[\Delta_0] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_{11}[\Delta] \arrow[d, tail, "\vartheta"] \\
\mathtt{C}_{110}[\Delta'_0] \arrow[r, two heads, "\mathtt{sh\backslash gc}"] & \mathtt{C}_{11}[\Delta']
\end{tikzcd}$$

So by taking $\mathtt{C}_{10} := \mathtt{C}_{110}[y\backslash t] \in \mathcal{C}^h_\vartheta$ we have:

$$\begin{tikzcd}
t_0 = \mathtt{C}_{110}[\Delta_0][y\backslash t] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_{11}[\Delta][y\backslash t] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = \mathtt{C}_{110}[\Delta'_0][y\backslash t] \arrow[r, two heads, "\mathtt{sh\backslash gc}"] & \mathtt{C}_{11}[\Delta'][y\backslash t] = t_3
\end{tikzcd}$$

(c) **The internal step $\mathtt{r}$ is to the right of $t_0 = t'_0[y\backslash r_0]$.** Then $\mathtt{r} : r_0 \to_{\mathtt{sh\backslash gc}} r$ and it is immediate to close the diagram:

$$\begin{tikzcd}
t_0 = \mathtt{C}_{11}[\Delta][y\backslash r_0] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_{11}[\Delta][y\backslash r] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = \mathtt{C}_{11}[\Delta'][y\backslash r_0] \arrow[r, two heads, "\mathtt{sh\backslash gc}"] & \mathtt{C}_{11}[\Delta'][y\backslash r] = t_3
\end{tikzcd}$$

4. **ESubLStr, $\mathtt{C}_1 = \mathtt{C}_{11}[y\backslash t]$ with $\mathtt{C}_{11} \in \mathcal{C}^h_{\vartheta\cup\{y\}}$ and $t \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$.** The situation is:

$$\begin{tikzcd}
t_0 \arrow[r, "\neg\vartheta"] & \mathtt{C}_{11}[\Delta][y\backslash t] = t_1 \arrow[d, tail, "\vartheta"] \\
& \mathtt{C}_{11}[\Delta'][y\backslash t] = t_3
\end{tikzcd}$$

We consider three cases, depending on whether (1) the internal step $\mathtt{r}$ is at the root of t_0, (2) t_0 is a substitution $t'_0[y\backslash r_0]$ and the step $\mathtt{r}$ is internal to t'_0, (3) t_0 is a substitution $t'_0[y\backslash r_0]$ and the step $\mathtt{r}$ is internal to r_0.

(a) **The internal step $\mathtt{r}$ is at the root of t_0.** Note that $\mathtt{r}$ cannot be a $\mathtt{dB}$, $\mathtt{fix}$ or $\mathtt{case}$ step, since it would be external. So it must be a $\mathtt{lsv}$ step of the form:

$$\mathtt{r} : t_0 = \mathtt{C}[\![z]\!][z\backslash v'\mathtt{L}'] \to^{\neg\vartheta}_{\mathtt{sh}} \mathtt{C}[\![v']\!][z\backslash v']\mathtt{L}' = t_1$$

Let $\mathtt{L}_1$ be a substitution context such that $\mathtt{L}_1[y\backslash t] = [z\backslash v']\mathtt{L}'$. Recall that $\Delta = \mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}]$. Using Lem. 96 let us strip $\mathtt{L}_1$ from $\mathtt{C}_{11}[\mathtt{C}_2[x\backslash v\mathtt{L}]]$. This gives us four possibilities, **A**, **B**, **C**, and **D** in the statement of Lem. 96.

i. **Case A.** Then:

$$\mathtt{C}_{11} = \mathtt{C}_{111}\mathtt{L}_1 \qquad \mathtt{C}[v'] = \mathtt{C}_{111}[\Delta]$$

where $\hat{\vartheta} = \mathsf{fz}^{\vartheta\cup\{y\}}(\mathtt{L}_1) \setminus \{y\}$ and $\mathtt{C}_{111} \in \mathcal{C}^h_{\hat{\vartheta}\cup\{y\}}$. We consider three cases, depending on whether the holes of $\mathtt{C}$ and $\mathtt{C}_{111}$ are disjoint, $\mathtt{C}$ is a prefix of $\mathtt{C}_{111}$, or $\mathtt{C}_{111}$ is a prefix of $\mathtt{C}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{111}$ are disjoint.** Then there is a two-hole context such that:

$$\widehat{\mathtt{C}}[\Box, v'] = \mathtt{C}_{111} \qquad \widehat{\mathtt{C}}[\Delta, \Box] = \mathtt{C}$$

By the fact that evaluation contexts are backwards-stable by substitutions (Lem. 89) there are two possibilities for $\widehat{\mathtt{C}}$: the left and the right branch of the disjunction. Let us analyze each branch:

- **Left branch.** Then $\widehat{\mathtt{C}}[\Box, z] \in \mathcal{C}^h_{\hat{\vartheta}\cup\{y\}}$. Note that $\hat{\vartheta} \cup \{y\} \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}'$ so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\widehat{\mathtt{C}}[\Box, z] \in \mathbb{X}_\vartheta$ and we may close the diagram as follows:

$$\begin{tikzcd}
t_0 = \widehat{\mathtt{C}}[\Delta, z][z\backslash v'\mathtt{L}'] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \widehat{\mathtt{C}}[\Delta, v'][z\backslash v']\mathtt{L}' = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = \widehat{\mathtt{C}}[\Delta', z][z\backslash v'\mathtt{L}'] \arrow[r, "\mathtt{sh}\backslash\mathtt{gc}"] & \widehat{\mathtt{C}}[\Delta', v'][z\backslash v']\mathtt{L}' = t_3
\end{tikzcd}$$

- **Right branch.** Then $\mathtt{C} \in \mathcal{C}^h_{\hat{\vartheta}\cup\{y\}}$. Note that $\hat{\vartheta} \cup \{y\} \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}'$ so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$. This contradicts the hypothesis that $\mathtt{r}$ is an internal step.

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{111}$.** Then by the decomposition of evaluation contexts lemma (Lem. 50) we know that $\mathtt{C} \in \mathcal{C}^h_{\hat{\vartheta}\cup\{y\}}$. Note that $\hat{\vartheta} \cup \{y\} \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}'$ so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C} \in \mathcal{C}^h_{\vartheta}$. This contradicts the hypothesis that $\mathtt{r}$ is an internal step.

C. **The context $\mathtt{C}_{111}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{111}[\mathtt{C}_1]$. So $\Delta = \mathtt{C}_1[v']$. Recall that $\Delta = \mathtt{C}_2[\![x]\!][x\backslash v\mathtt{L}]$.
The remainder of this case is analogous to case 3(a)iC, by case analysis on whether the hole of $\mathtt{C}_1$ lies to the left or inside the substitution $[x\backslash v\mathtt{L}]$.

ii. **Case B.** Then:

$$\mathtt{C}_{11} = \mathtt{C}_{111}[\![w]\!]\mathscr{L}\{\Box\} \qquad \mathtt{C}[v'] = \mathtt{C}_{111}[\![w]\!]$$
$$\mathtt{L}_1 = \Box\mathscr{L}\{\Delta\}$$

where $\hat{\vartheta} = \mathsf{fz}^{\vartheta\cup\{y\}}(\mathtt{L}_1) \setminus \{y\}$, the evaluation context $\mathtt{C}_{111}$ is in $\mathcal{C}^h_{\hat{\vartheta}\cup\{y\}}$, and $\mathscr{L}$ is a $(\vartheta \cup \{y\}, w)$-chain context.
We consider three further subcases, depending on the position of the hole of $\mathtt{C}$ relative to the position of the hole of $\mathtt{C}_{111}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{111}$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\Box, v'] = \mathtt{C}_{111} \qquad \widehat{\mathtt{C}}[w, \Box] = \mathtt{C}$$

By the fact that evaluation contexts are backwards-stable by substitutions (Lem. 89) there are two possibilities for $\widehat{\mathtt{C}}$: the left and the right branch of the disjunction. Let us analyze each branch:

- **Left branch.** Then $\widehat{\mathtt{C}}[\Box, z] \in \mathcal{C}^h_{\hat{\vartheta}\cup\{y\}}$. Note that $\hat{\vartheta} \cup \{y\} \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}'$, so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we obtain that $\widehat{\mathtt{C}}[\Box, z] \in \mathcal{C}^h_{\vartheta}$. This allows us to close the diagram as follows:

$$\begin{array}{ccc} t_0 = \widehat{\mathtt{C}}[\Delta, z][z\backslash v'\mathtt{L}'] & \xrightarrow{\neg\vartheta} & \widehat{\mathtt{C}}[\Delta, v'][z\backslash v']\mathtt{L}' = t_1 \\ \downarrow{\scriptstyle\vartheta} & & \downarrow{\scriptstyle\vartheta} \\ t_2 = \widehat{\mathtt{C}}[\Delta', z][z\backslash v'\mathtt{L}'] & \xrightarrow{\mathtt{sh\backslash gc}} & \widehat{\mathtt{C}}[\Delta', v'][z\backslash v']\mathtt{L}' = t_3 \end{array}$$

- **Right branch.** Then $\mathtt{C} \in \mathcal{C}^h_{\hat{\vartheta}\cup\{y\}}$. Note that $\hat{\vartheta} \cup \{y\} \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}'$, so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we obtain that $\widehat{\mathtt{C}}[\Box, z] \in \mathcal{C}^h_{\vartheta}$. This contradicts the fact that the step $\mathtt{r}$ is internal.

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{111}$.** Then by the decomposition lemma for evaluation contexts (Lem. 50) we know that $\mathtt{C} \in \mathcal{C}^h_{\hat{\vartheta}\cup\{y\}}$. Note that $\hat{\vartheta} \cup \{y\} \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}'$, so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we obtain that $\widehat{\mathtt{C}}[\Box, z] \in \mathcal{C}^h_{\vartheta}$. This contradicts the fact that the step $\mathtt{r}$ is internal.

C. **The context $\mathtt{C}_{111}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{111}[\mathtt{C}_1]$. Hence $w = \mathtt{C}_1[v]$, which is impossible.

iii. **Case C.** Then $\mathtt{C}_{11}$ is a substitution context, and:

$$\mathtt{C}_2 = \mathtt{C}_{21}\mathtt{L}_2 \qquad \mathtt{L}_1 = \mathtt{C}_{11}[\mathtt{L}_2[x\backslash v\mathtt{L}]] \qquad \mathtt{C}[v'] = \mathtt{C}_{21}[\![x]\!]$$

where $\hat{\vartheta}' = \mathtt{fz}^{\vartheta'}(\mathtt{L}_2)$ and the evaluation context $\mathtt{C}_{21}$ is in $\mathcal{C}^{h'}_{\hat{\vartheta}'}$.
The remainder of this case is analogous to case 3(a)iii, by case analysis on the relative positions of the holes of $\mathtt{C}$ and $\mathtt{C}_{21}$.

iv. **Case D.** Then $\mathtt{C}_{11}$ is a substitution context, and:

$$\mathtt{C}_2 = \mathtt{C}_{21}[\![w]\!]\mathscr{L}\{\Box\} \qquad \mathtt{L}_1 = \mathtt{C}_{11}[\Box\mathscr{L}\{x\}[x\backslash v\mathtt{L}]]$$
$$\mathtt{C}[v'] = \mathtt{C}_{21}[\![w]\!]$$

where $\hat{\vartheta}' = \mathtt{fz}^{\vartheta\cup\{y\}}(\mathtt{L}_1)$, the evaluation context $\mathtt{C}_{21}$ is in $Y^{\hat{\vartheta}'}$, and $\mathscr{L}$ is a (ϑ', w)-chain context.
The remainder of this case is analogous to case 3(a)iv, by case analysis on the relative positions of the holes of $\mathtt{C}$ and $\mathtt{C}_{21}$.

(b) **The internal step $\mathtt{r}$ is to the left of $t_0 = t'_0[y\backslash r_0]$.** Then there is a step $\mathtt{r}_1 : t'_0 \to_{\mathtt{sh\backslash gc}} \mathtt{C}_{11}[\Delta][y\backslash t]$. It must be a $(\vartheta \cup \{y\})$-internal step, for otherwise $\mathtt{r}$ would be ϑ-external.
By *i.h.* we have that there exists an evaluation context $\mathtt{C}_{110} \in \mathcal{C}^h_{\vartheta}$, a $\mathtt{lsv}$ redex Δ_0 and

Δ_0' its contractum such that $t_0' = \mathtt{C}_{110}[\Delta_0]$ and:

$$\begin{tikzcd}
\mathtt{C}_{110}[\Delta_0] \arrow[r, "\neg(\vartheta\cup\{y\})"] \arrow[d, tail, "(\vartheta\cup\{y\})"'] & \mathtt{C}_{11}[\Delta] \arrow[d, tail, "(\vartheta\cup\{y\})-\mathrm{need}"] \\
\mathtt{C}_{110}[\Delta_0'] \arrow[r, two heads, "\mathtt{sh}\backslash\mathtt{gc}"] & \mathtt{C}_{11}[\Delta']
\end{tikzcd}$$

So by taking $\mathtt{C}_{10} := \mathtt{C}_{110}[y\backslash t] \in \mathcal{C}_\vartheta^h$ we have:

$$\begin{tikzcd}
t_0 = \mathtt{C}_{110}[\Delta_0][y\backslash r] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_{11}[\Delta][y\backslash r] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = \mathtt{C}_{110}[\Delta_0'][y\backslash r] \arrow[r, two heads, "\mathtt{sh}\backslash\mathtt{gc}"] & \mathtt{C}_{11}[\Delta'][y\backslash r] = t_3
\end{tikzcd}$$

(c) **The internal step $\mathtt{r}$ is to the right of $t_0 = t_0'[y\backslash r_0]$.** Then the internal step $\mathtt{r}$ is of the form:

$$\mathtt{C}_1[\Delta][y\backslash r_0] \to_{\mathtt{sh}}^{\neg\vartheta} \mathtt{C}_1[\Delta][y\backslash t]$$

and there is a step $\mathtt{r}_1 : r_0 \to_{\mathtt{sh}\backslash\mathtt{gc}} t$. We consider two cases, depending on whether y is a structural variable in $\mathtt{C}_1$.

i. **If $y \in \mathsf{sv}(\mathtt{C}_1)$.** Then by the fact that structural variables are below evaluation contexts (Lem. 88) we have that there is a context $\mathtt{C}_3 \in \mathcal{C}_\vartheta^h$ such that $\mathtt{C}_1[\Delta] = \mathtt{C}_3[\![y]\!]$. Let us consider two further subcases, depending on whether $\mathtt{r}_1$ is ϑ-external or ϑ-internal:

A. **If $\mathtt{r}_1$ is a ϑ-external step.** Then the step

$$\mathtt{r} : \mathtt{C}_1[\Delta][y\backslash r_0] = \mathtt{C}_3[\![y]\!][y\backslash r_0] \to_{\mathtt{sh}\backslash\mathtt{gc}} \mathtt{C}_3[\![y]\!][y\backslash t]$$

is ϑ-external, contradicting the hypothesis that it is ϑ-internal.

B. **If $\mathtt{r}_1$ is a ϑ-internal step.** Then by the fact that normal forms are backwards preserved by internal steps (Lem. 104) we have that r_0 is a structure in $\mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$, so $\mathtt{C}_1[y\backslash r_0]$ is an evaluation context in $\mathcal{C}_\vartheta^h$ and we may close the diagram as follows:

$$\begin{tikzcd}
t_0 = \mathtt{C}_1[\Delta][y\backslash r_0] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_1[\Delta][y\backslash t] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = \mathtt{C}_1[\Delta'][y\backslash r_0] \arrow[r, "\mathtt{sh}\backslash\mathtt{gc}"] & \mathtt{C}_1[\Delta'][y\backslash t] = t_3
\end{tikzcd}$$

ii. **If $y \notin \mathsf{sv}(\mathtt{C}_1)$.** Then by the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C}$ is an evaluation context in $\mathcal{C}_\vartheta^h$. Hence, regardless of whether r_0 is a structure or not a structure, the context $\mathtt{C}_1[x\backslash r_0]$ is an evaluation context in $\mathcal{C}_\vartheta^h$. It is then straightforward to close the diagram, as in the previous case.

5. **ESubsR,** $\mathtt{C}_1 = \mathtt{C}_{11}[\![y]\!][y\backslash\mathtt{C}_3]$ **with** $\mathtt{C}_{11} \in \mathcal{C}^h_\vartheta$ **and** $\mathtt{C}_3 \in \mathcal{C}^{\cdot}_\vartheta$**.** The situation is:

$$\begin{array}{ccc} t_0 & \xrightarrow{\neg\vartheta} & \mathtt{C}_{11}[\![y]\!][y\backslash\mathtt{C}_3[\Delta]] = t_1 \\ & & \downarrow \vartheta \\ & & \mathtt{C}_{11}[\![y]\!][y\backslash\mathtt{C}_3[\Delta']] = t_3 \end{array}$$

We consider three cases, depending on whether (1) the internal step $\mathtt{r}$ is at the root of t_0, (2) t_0 is a substitution $t_0'[y\backslash r_0]$ and the step $\mathtt{r}$ is internal to t_0', (3) t_0 is a substitution $t_0'[y\backslash r_0]$ and the step $\mathtt{r}$ is internal to r_0.

(a) **The internal step $\mathtt{r}$ is at the root of t_0.** Note that $\mathtt{r}$ cannot be a $\mathtt{dB}$, $\mathtt{fix}$ or $\mathtt{case}$ step, since it would be external, so it must be a $\mathtt{lsv}$ step of the form:

$$\mathtt{r} : t_0 = \mathtt{C}[\![z]\!][z\backslash v'\mathtt{L}'] \to^{\neg\vartheta}_{\mathtt{sh}} \mathtt{C}[v'][z\backslash v']\mathtt{L}' = t_1$$

Let $\mathtt{L}_1$ be a substitution context such that $\mathtt{L}_1[y\backslash\mathtt{C}_1[\Delta]] = [z\backslash v']\mathtt{L}'$. Using Lem. 95 let us strip $\mathtt{L}_1$ from $\mathtt{C}_{11}[y\backslash\mathtt{C}_1[\Delta]]$. This gives us two possibilities, **A** and **B** in the statement of Lem. 95.

i. **Case A.** Then:

$$\mathtt{C}_{11} = \mathtt{C}_{111}\mathtt{L}_1 \qquad \mathtt{C}[v'] = \mathtt{C}_{111}[\![y]\!]$$

where $\hat{\vartheta} = \mathsf{fz}^\vartheta(\mathtt{L}_1)$ and $\mathtt{C}_{111} \in \mathcal{C}^{h'}_{\hat{\vartheta}}$. We consider three cases, depending on whether the holes of $\mathtt{C}$ and $\mathtt{C}_{111}$ are disjoint, $\mathtt{C}$ is a prefix of $\mathtt{C}_{111}$, or $\mathtt{C}_{111}$ is a prefix of $\mathtt{C}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{111}$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\Box, v'] = \mathtt{C}_{111} \qquad \widehat{\mathtt{C}}[y, \Box] = \mathtt{C}$$

Note that the internal step $\mathtt{r}$ is of the form:

$$\mathtt{r} : \widehat{\mathtt{C}}[y, z][z\backslash v'\mathtt{L}'] \to^{\neg\vartheta}_{\mathtt{sh}} \widehat{\mathtt{C}}[y, v'][z\backslash v']\mathtt{L}'$$

and y is bound by $\mathtt{L}_1[y\backslash\mathtt{C}_1[\Delta]] = [z\backslash v']\mathtt{L}'$ on the right-hand side, so it must be the case that $y = z$, for otherwise y would be free on the left-hand side, and free variables cannot become bound.
Hence, since $y = z$, we have that $v' = \mathtt{C}_1[\Delta]$. This is impossible, since answers do not have redexes below non-answer evaluation contexts (Lem. 97).

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{111}$.** Then by the decomposition of evaluation contexts lemma (Lem. 50) we know that $\mathtt{C} \in \mathcal{C}^{h'}_{\hat{\vartheta}}$. Note that $\hat{\vartheta} \cup \{y\} \subseteq \vartheta \cup \mathrm{dom}\,\mathtt{L}'$, so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we conclude that $\mathtt{C} \in \mathcal{C}^h_\vartheta$. This contradicts the fact that $\mathtt{r}$ is a ϑ-internal step.

C. **The context of $\mathtt{C}_{111}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{111}[\mathtt{C}_1]$, so $y = \mathtt{C}_1[v']$ which is impossible.

ii. **Case B.** Then:

$$\mathtt{C}_{11} = \mathtt{C}_{111}[\![w]\!]\mathscr{L}\{\Box\} \qquad \mathtt{C}[v'] = \mathtt{C}_{111}[\![w]\!]$$
$$\mathtt{L}_1 = \Box\mathscr{L}\{y\}$$

where $\hat{\vartheta} = \mathsf{fz}^{\vartheta}(\mathtt{L}_1)$, the evaluation context $\mathtt{C}_{111}$ is in $\mathcal{C}^{h'}_{\hat{\vartheta}}$, and $\mathscr{L}$ is a (ϑ, w)-chain context.

We consider three cases, depending on whether the holes of $\mathtt{C}$ and $\mathtt{C}_{111}$ are disjoint, $\mathtt{C}$ is a prefix of $\mathtt{C}_{111}$, or $\mathtt{C}_{111}$ is a prefix of $\mathtt{C}$.

A. **The hole of $\mathtt{C}$ and the hole of $\mathtt{C}_{111}$ are disjoint.** Then there is a two-hole context $\widehat{\mathtt{C}}$ such that:

$$\widehat{\mathtt{C}}[\Box, v'] = \mathtt{C}_{111} \qquad \widehat{\mathtt{C}}[w, \Box] = \mathtt{C}$$

The internal step $\mathtt{r}$ is then of the form:

$$\mathtt{r} : \widehat{\mathtt{C}}[w, z][z \backslash v'\mathtt{L}'] \to^{\neg\vartheta}_{\mathtt{sh}} \widehat{\mathtt{C}}[w, v'][z \backslash v']\mathtt{L}'$$

Note that w is bound by $[z\backslash v']\mathtt{L}' = \mathscr{L}\{y\}[y\backslash v\mathtt{L}]$ on the right-hand side, hence $w = z$, since otherwise w would be free on the left-hand side, and free variables cannot become bound.

Consider the term p such that w is bound to p in the context $\Box\mathscr{L}\{y\}[y\backslash \mathtt{C}_1[\Delta]]$. There are two possibilities: either $\mathscr{L}$ has no jumps and $y = w$ with $p = \mathtt{C}_1[\Delta]$, or $\mathscr{L}$ has at least one jump and p is of the form $\mathtt{I}^{\vartheta_1}_1[\![w_1]\!]$ for some non-answer evaluation context $\mathtt{I}^{\vartheta_1} \in \mathsf{NACtxt}_{\vartheta_1}$. Since $w = z$, we have that $p = v'$. In any case, this is impossible by the fact that answers do not have redexes or variables below non-answer evaluation contexts (Lem. 97).

B. **The context $\mathtt{C}$ is a prefix of $\mathtt{C}_{111}$.** Then by the decomposition of evaluation contexts lemma (Lem. 50) we know that $\mathtt{C} \in \mathcal{C}^{h'}_{\hat{\vartheta}}$. Note that $\hat{\vartheta} \subseteq \vartheta \cup \operatorname{dom} \mathtt{L}'$, so by repeatedly applying the fact that non-structural variables are not required in "ϑ" (Lem. 87) we have that $\mathtt{C} \in \mathcal{C}^{h}_{\vartheta}$, contradicting the fact that the step $\mathtt{r}$ is ϑ-internal.

C. **The context $\mathtt{C}_{111}$ is a prefix of $\mathtt{C}$.** Then $\mathtt{C} = \mathtt{C}_{111}[\mathtt{C}_1]$ so $w = \mathtt{C}_1[v']$, which is impossible.

(b) **The internal step $\mathtt{r}$ is to the left of $t_0 = t'_0[y\backslash r_0]$.** Let $\mathtt{r}_1 : t'_0 \to_{\mathtt{sh}\backslash\mathtt{gc}} \mathtt{C}_{11}[\![y]\!]$ be the step isomorphic to $\mathtt{r}$ but going under the substitution $[y\backslash \mathtt{C}_1[\Delta]]$. Note that $\mathtt{r}_1$ cannot be ϑ-external since, by the fact that adding an arbitrary substitution to a ϑ-evaluation context yields a ϑ-evaluation context (Lem. 80) this would imply that $\mathtt{r}$ is also ϑ-external. So $\mathtt{r}_1$ is ϑ-internal and we may apply the fact that needed variables are backwards preserved by internal steps (Lem. 105) to conclude that t'_0 has to be of the form $\mathtt{C}_{110}[\![y]\!]$

So by taking $\mathtt{C}_{10} := \mathtt{C}_{110}[\![y]\!][y\backslash \mathtt{C}_3]$ we may close the diagram as follows:

$$\begin{tikzcd}
t_0 = \mathtt{C}_{110}[\![y]\!][y\backslash \mathtt{C}_3[\Delta]] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_{11}[\![y]\!][y\backslash \mathtt{C}_3[\Delta]] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = \mathtt{C}_{110}[\![y]\!][y\backslash \mathtt{C}_3[\Delta']] \arrow[r, "\mathtt{sh}\backslash\mathtt{gc}"] & \mathtt{C}_{11}[\![y]\!][y\backslash \mathtt{C}_3[\Delta']] = t_3
\end{tikzcd}$$

(c) **The internal step $\mathtt{r}$ is to the right of $t_0 = t'_0[y\backslash r_0]$.** Let $\mathtt{r}_1 : r_0 \to_{\mathtt{sh\backslash gc}} \mathtt{C}_3[\Delta]$ be the step isomorphic to $\mathtt{r}$ but going inside the substitution $\mathtt{C}_{11}[\![y]\!][y\backslash\Box]$. Note that $\mathtt{r}_1$ cannot be ϑ-external since this would imply that $\mathtt{r}$ is ϑ-external.

By *i.h.* we have that there exists a non-answer evaluation context $\mathtt{C}_{30} \in \mathcal{C}_{\vartheta}^{\cdot}$, a $\mathtt{lsv}$ redex Δ_0 and Δ'_0 its contractum such that $r_0 = \mathtt{C}_{30}[\Delta_0]$ and:

$$\begin{tikzcd}
\mathtt{C}_{30}[\Delta_0] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_3[\Delta] \arrow[d, tail, "\vartheta"] \\
\mathtt{C}_{30}[\Delta'_0] \arrow[r, two heads, "\mathtt{sh\backslash gc}"] & \mathtt{C}_3[\Delta']
\end{tikzcd}$$

So by taking $\mathtt{C}_{10} := \mathtt{C}_{11}[\![y]\!][y\backslash\mathtt{C}_{30}]$ we have:

$$\begin{tikzcd}
t_0 = \mathtt{C}_{11}[\![y]\!][y\backslash\mathtt{C}_{30}[\Delta_0]] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_{11}[\![y]\!][y\backslash\mathtt{C}_3[\Delta]] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = \mathtt{C}_{11}[\![y]\!][y\backslash\mathtt{C}_{30}[\Delta'_0]] \arrow[r, two heads, "\mathtt{sh\backslash gc}"] & \mathtt{C}_{11}[\![y]\!][y\backslash\mathtt{C}_3[\Delta']] = t_3
\end{tikzcd}$$

6. **EAppRStr, $\mathtt{C}_1 = p\,\mathtt{C}_{11}$, where $p \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$ and $\mathtt{C}_{11} \in \mathcal{C}_{\vartheta}^{h}$.** The situation is:

$$\begin{tikzcd}
t_0 \arrow[r, "\neg\vartheta"] & p\,\mathtt{C}_{11}[\Delta] = t_1 \arrow[d, tail, "\vartheta"] \\
 & p\,\mathtt{C}_{11}[\Delta'] = t_3
\end{tikzcd}$$

Note that the internal step $\mathtt{r}$ cannot be at the root: it cannot be a $\mathtt{dB}$, $\mathtt{fix}$ or $\mathtt{case}$ step, since it would be external, and it cannot be a $\mathtt{lsv}$ step, since then there would be a substitution node at the root of t_1.

So t_0 must be an application node $r_1\,r_2$ and there are two remaining cases: (1) the step $\mathtt{r}$ is internal to r_1, (2) the step $\mathtt{r}$ is internal to r_2.

(a) **The internal step $\mathtt{r}$ is internal to the left of $t_0 = r_1\,r_2$.** Then $t_0 = r_1\,\mathtt{C}_{11}[\Delta]$. Let $\mathtt{r}_1 : r_1 \to_{\mathtt{sh\backslash gc}} p$ be the step isomorphic to $\mathtt{r}$ below the context $\Box\,\mathtt{C}_{11}[\Delta]$. Note that $\mathtt{r}_1$ cannot be ϑ-external as this would imply that $\mathtt{r}$ is also ϑ-external. Hence $\mathtt{r}_1$ is ϑ-internal.

By the fact that strong normal forms are backwards stable by internal steps (Lem. 104) we know that r_1 must be a strong ϑ-structure, *i.e.* $r_1 \in \mathcal{S}_\vartheta \cup \mathcal{E}_\vartheta$. By taking $\mathtt{C}_{10} := r_1\,\mathtt{C}_{11}$ we may close the diagram as follows:

$$\begin{tikzcd}
t_0 = r_1\,\mathtt{C}_{11}[\Delta] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & p\,\mathtt{C}_{11}[\Delta] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = r_1\,\mathtt{C}_{11}[\Delta'] \arrow[r, "\mathtt{sh\backslash gc}"] & p\,\mathtt{C}_{11}[\Delta'] = t_3
\end{tikzcd}$$

(b) **The internal step $\mathtt{r}$ is internal to the right of $t_0 = r_1\, r_2$.** Then $t_0 = p\, r_2$. Let $\mathtt{r}_1 : r_2 \to_{\mathtt{sh\backslash gc}} \mathtt{C}_{11}[\Delta]$ be the step isomorphic to $\mathtt{r}$ below the context $p\,\Box$. Note that $\mathtt{r}_1$ cannot ϑ-external since this would imply that $\mathtt{r}$ is also ϑ-external. Hence $\mathtt{r}_1$ is ϑ-internal. By *i.h.* we have that there exists an evaluation context $\mathtt{C}_{110} \in \mathcal{C}^h_\vartheta$, a $\mathtt{lsv}$ redex Δ_0 and Δ'_0 its contractum such that $r_2 = \mathtt{C}_{110}[\Delta_0]$ and:

$$\begin{tikzcd}
\mathtt{C}_{110}[\Delta_0] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & \mathtt{C}_{11}[\Delta] \arrow[d, tail, "\vartheta"] \\
\mathtt{C}_{110}[\Delta'_0] \arrow[r, "\mathtt{sh\backslash gc}"] & \mathtt{C}_{11}[\Delta']
\end{tikzcd}$$

We may close the diagram by taking $\mathtt{C}_{10} := p\,\mathtt{C}_{110}$:

$$\begin{tikzcd}
t_0 = p\,\mathtt{C}_{110}[\Delta_0] \arrow[r, "\neg\vartheta"] \arrow[d, tail, "\vartheta"'] & p\,\mathtt{C}_{11}[\Delta] = t_1 \arrow[d, tail, "\vartheta"] \\
t_2 = p\,\mathtt{C}_{110}[\Delta'_0] \arrow[r, "\mathtt{sh\backslash gc}"] & p\,\mathtt{C}_{11}[\Delta'] = t_3
\end{tikzcd}$$

7. **ELam, $\mathtt{C}_1 = \lambda y.\mathtt{C}$, where $\mathtt{C} \in \mathcal{C}^h_\vartheta$.** Straightforward by *i.h.*, as in the case for the ELam rule of the previous lemma (Lem. 106, case 8).

8. **EAppCons, $\mathtt{C} = r\,\mathtt{C}_1$, where $r \in \mathcal{K}_\vartheta$, $h = \mathsf{hc}(r)$, and $\mathtt{C}_1 \in \mathcal{C}^h_\vartheta$.** Similar to EAppRStr.

9. **ECase1.** Similar to previous cases.

10. **ECase2.** Similar to previous cases.

□

Lemma 108 (Permutation of internal steps and external $\mathtt{fix}$ steps)**.** *Given any set of variables ϑ such that $\mathsf{fv}(t_0) \subseteq \vartheta$, if $t_0 \to^{\neg\vartheta}_{\mathtt{sh}} t_1 \rightarrowtail^{\vartheta}_{\mathtt{sh}} t_3$ where the second step is a $\mathtt{fix}$ step, there exists a term t_2 such that $t_0 \rightarrowtail^{\vartheta}_{\mathtt{sh}} t_2 \twoheadrightarrow_{\mathtt{sh\backslash gc}} t_3$ where the first step is a $\mathtt{fix}$ step. An explicit construction for the diagrams is given.*

Proof. Similar to Lem. 106. □

Lemma 109 (Permutation of internal steps and external $\mathtt{case}$ steps)**.** *Given any set of variables ϑ such that $\mathsf{fv}(t_0) \subseteq \vartheta$, if $t_0 \to^{\neg\vartheta}_{\mathtt{sh}} t_1 \rightarrowtail^{\vartheta}_{\mathtt{sh}} t_3$ where the second step is a $\mathtt{case}$ step, there exists a term t_2 such that $t_0 \rightarrowtail^{\vartheta}_{\mathtt{sh}} t_2 \twoheadrightarrow_{\mathtt{sh\backslash gc}} t_3$ where the first step is a $\mathtt{case}$ step. An explicit construction for the diagrams is given.*

Proof. Similar to Lem. 106 □

Lemma 110 (Permutation of internal/external steps)**.** *Given any set of variables ϑ such that $\mathsf{fv}(t_0) \subseteq \vartheta$, if $t_0 \to^{\neg\vartheta}_{\mathtt{sh}} t_1 \rightarrowtail^{\vartheta}_{\mathtt{sh}} t_3$ there exists a term t_2 such that $t_0 \rightarrowtail^{\vartheta}_{\mathtt{sh}} t_2 \twoheadrightarrow_{\mathtt{sh\backslash gc}} t_3$. More precisely, the diagram can be closed constructively and exactly the following swaps are allowed:*

$$\to^{\neg\vartheta}_{\mathtt{sh,lsv}} \rightarrowtail^{\vartheta}_{\mathtt{sh,lsv}} \quad\subseteq\quad (\rightarrowtail^{\vartheta}_{\mathtt{sh,lsv}})^+ (\to^{\neg\vartheta}_{\mathtt{sh,lsv}})^*$$

and also:

- $\mathtt{dB}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,dB}}\rightarrowtail^{\vartheta}_{\mathtt{sh,dB}} \quad\subseteq\quad (\rightarrowtail^{\vartheta}_{\mathtt{sh,dB}})^{+}(\to^{\neg\vartheta}_{\mathtt{sh,dB}})^{*}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,dB}}\rightarrowtail^{\vartheta}_{\mathtt{sh,fix}} \quad\subseteq\quad (\rightarrowtail^{\vartheta}_{\mathtt{sh,fix}})^{+}(\to^{\neg\vartheta}_{\mathtt{sh,dB}})^{*}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,dB}}\rightarrowtail^{\vartheta}_{\mathtt{sh,case}} \quad\subseteq\quad (\rightarrowtail^{\vartheta}_{\mathtt{sh,case}})^{+}(\to^{\neg\vartheta}_{\mathtt{sh,dB}})^{*}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,lsv}}\rightarrowtail^{\vartheta}_{\mathtt{sh,dB}} \quad\subseteq\quad \rightarrowtail^{\vartheta}_{\mathtt{sh,dB}}(\to^{\neg\vartheta}_{\mathtt{sh}})^{*}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,dB}}\rightarrowtail^{\vartheta}_{\mathtt{sh,lsv}} \quad\subseteq\quad \rightarrowtail^{\vartheta}_{\mathtt{sh,lsv}}(\to^{\neg\vartheta}_{\mathtt{sh}})^{*}$
- $\mathtt{fix}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,fix}}\rightarrowtail^{\vartheta}_{\mathtt{sh,fix}} \quad\subseteq\quad (\rightarrowtail^{\vartheta}_{\mathtt{sh,fix}})^{+}(\to^{\neg\vartheta}_{\mathtt{sh,fix}})^{*}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,fix}}\rightarrowtail^{\vartheta}_{\mathtt{sh,dB}} \quad\subseteq\quad (\rightarrowtail^{\vartheta}_{\mathtt{sh,dB}})^{+}(\to^{\neg\vartheta}_{\mathtt{sh,fix}})^{*}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,fix}}\rightarrowtail^{\vartheta}_{\mathtt{sh,case}} \quad\subseteq\quad (\rightarrowtail^{\vartheta}_{\mathtt{sh,case}})^{+}(\to^{\neg\vartheta}_{\mathtt{sh,fix}})^{*}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,lsv}}\rightarrowtail^{\vartheta}_{\mathtt{sh,fix}} \quad\subseteq\quad \rightarrowtail^{\vartheta}_{\mathtt{sh,fix}}(\to^{\neg\vartheta}_{\mathtt{sh}})^{*}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,fix}}\rightarrowtail^{\vartheta}_{\mathtt{sh,lsv}} \quad\subseteq\quad \rightarrowtail^{\vartheta}_{\mathtt{sh,lsv}}(\to^{\neg\vartheta}_{\mathtt{sh}})^{*}$
- $\mathtt{case}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,case}}\rightarrowtail^{\vartheta}_{\mathtt{sh,case}} \quad\subseteq\quad (\rightarrowtail^{\vartheta}_{\mathtt{sh,case}})^{+}(\to^{\neg\vartheta}_{\mathtt{sh,case}})^{*}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,case}}\rightarrowtail^{\vartheta}_{\mathtt{sh,dB}} \quad\subseteq\quad (\rightarrowtail^{\vartheta}_{\mathtt{sh,dB}})^{+}(\to^{\neg\vartheta}_{\mathtt{sh,case}})^{*}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,case}}\rightarrowtail^{\vartheta}_{\mathtt{sh,fix}} \quad\subseteq\quad (\rightarrowtail^{\vartheta}_{\mathtt{sh,fix}})^{+}(\to^{\neg\vartheta}_{\mathtt{sh,case}})^{*}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,lsv}}\rightarrowtail^{\vartheta}_{\mathtt{sh,case}} \quad\subseteq\quad \rightarrowtail^{\vartheta}_{\mathtt{sh,case}}(\to^{\neg\vartheta}_{\mathtt{sh}})^{*}$
  - $\to^{\neg\vartheta}_{\mathtt{sh,case}}\rightarrowtail^{\vartheta}_{\mathtt{sh,lsv}} \quad\subseteq\quad \rightarrowtail^{\vartheta}_{\mathtt{sh,lsv}}(\to^{\neg\vartheta}_{\mathtt{sh}})^{*}$

Proof. Let us call $\mathtt{r}$ to the internal step $t_0 \to^{\neg\vartheta}_{\mathtt{sh}} t_1$ and $\mathtt{r}'$ to the external step $t_1 \rightarrowtail^{\vartheta}_{\mathtt{sh}} t_3$. The proof goes by case analysis on the kind of step $\mathtt{r}'$. If $\mathtt{r}'$ is a $\mathtt{dB}$ step, this is a consequence of Lem. 106. If $\mathtt{r}'$ is a $\mathtt{lsv}$ step, this is a consequence of Lem. 107. If $\mathtt{r}'$ is a $\mathtt{fix}$ step, this is a consequence of Lem. 108. If $\mathtt{r}'$ is a $\mathtt{case}$ step, this is a consequence of Lem. 109. Note that in all cases the construction is given inductively. In all the base cases, the diagram is closed according to the allowed swaps. In all the inductive cases, the diagram is closed using the same kind of swaps as in the inductive hypothesis. $\square$

Definition 111 (Square Factorization System)**.** *A square factorization system (SFS) is a set S and four reduction relations $(\rightsquigarrow_{\bullet}, \rightsquigarrow_{\circ}, \mapsto_{\bullet}, \mapsto_{\circ})$ on S s.t. the following conditions hold:*

1. *Termination: $\rightsquigarrow_{\circ}$ and $\mapsto_{\circ}$ strongly normalizing.*
2. *Row-swap 1: $\rightsquigarrow_{\bullet}\rightsquigarrow_{\circ}\subseteq\rightsquigarrow_{\circ}^{+}\rightsquigarrow_{\bullet}^{*}$*
3. *Row-swap 2: $\mapsto_{\bullet}\mapsto_{\circ}\subseteq\mapsto_{\circ}^{+}\mapsto_{\bullet}^{*}$*
4. *Diagonal-swap 1: $\mapsto_{\bullet}\rightsquigarrow_{\circ}\subseteq\rightsquigarrow_{\circ}\mapsto^{*}$*

5. *Diagonal-swap 2:* $\rightsquigarrow_{\bullet}\mapsto_{\circ}\subseteq\mapsto_{\circ}\rightsquigarrow^{*}$

where $\rightsquigarrow=\rightsquigarrow_{\circ}\cup\rightsquigarrow_{\bullet}$ *and* $\mapsto=\mapsto_{\bullet}\cup\mapsto_{\circ}$.

Lemma 112 (Factorization for SFS (Thm 5.2(2) in [Acc12]))**.** *Consider a SFS* $(\rightsquigarrow_{\bullet},\rightsquigarrow_{\circ},\mapsto_{\bullet},\mapsto_{\circ})$ *on* S*. Let* $\to:=\rightsquigarrow\cup\mapsto$ *and* $\rightsquigarrow:=\rightsquigarrow_{\bullet}\cup\rightsquigarrow_{\circ}$ *and* $\mapsto:=\mapsto_{\bullet}\cup\mapsto_{\circ}$ *and* $\to_{\circ}:=\rightsquigarrow_{\circ}\cup\mapsto_{\circ}$ *and* $\to_{\bullet}:=\rightsquigarrow_{\bullet}\cup\mapsto_{\bullet}$*. Then* $\to^{*}\subseteq\to_{\circ}^{*}\to_{\bullet}^{*}$

Lemma 113 (Postponement of internal steps)**.** *If* $t \twoheadrightarrow_{\mathtt{sh}\backslash\mathtt{gc}} s$ *such that* s *is in* $\to_{\mathtt{sh}\backslash\mathtt{gc}}$*-normal form, given any set of variables* ϑ *such that* $\mathsf{fv}(t)\subseteq\vartheta$*, there is a term* u *in* $\rightarrowtail^{\vartheta}_{\mathtt{sh}}$*-normal form such that* $t \rightarrowtail\!\!\twoheadrightarrow^{\vartheta}_{\mathtt{sh}} u \twoheadrightarrow^{\neg\vartheta}_{\mathtt{sh}} s$*.*
Diagramatically:

$$\begin{tikzcd}
\Lambda_{\mathtt{sh}} \arrow[r, two heads, "\mathtt{sh}\backslash\mathtt{gc}"] \arrow[d, tail, two heads, "\vartheta"'] & \mathsf{NF}(\to_{\mathtt{sh}\backslash\mathtt{gc}}) \\
\mathsf{NF}(\rightarrowtail^{\vartheta}_{\mathtt{sh}}) \arrow[ur, dotted, two heads, "\neg\vartheta"'] &
\end{tikzcd}$$

Proof. This is an immediate consequence Lem. 112 where

- $\rightsquigarrow_{\circ}:=\rightarrowtail^{\vartheta}_{\mathtt{sh,dB}}\cup\rightarrowtail^{\vartheta}_{\mathtt{sh,fix}}\cup\rightarrowtail^{\vartheta}_{\mathtt{sh,case}}$
- $\mapsto_{\circ}:=\rightarrowtail^{\vartheta}_{\mathtt{sh,lsv}}$
- $\rightsquigarrow_{\bullet}:=\to^{\neg\vartheta}_{\mathtt{sh,dB}}\cup\to^{\neg\vartheta}_{\mathtt{sh,fix}}\cup\to^{\neg\vartheta}_{\mathtt{sh,case}}$
- $\mapsto_{\bullet}:=\to^{\neg\vartheta}_{\mathtt{sh,lsv}}$

$\square$